

Engineering Principles 1



Getting to know your unit

Assessment

This unit is externally assessed using an unseen paper-based examination that is marked by Pearson.

To make an effective contribution to the design and development of engineered products and systems, you must be able to draw on the principles laid down by the pioneers of engineering science. The theories developed by the likes of Newton and Ohm are at the heart of the work carried out by today's multi-skilled engineering workforce. This unit covers a range of both mechanical and electrical principles and some of the necessary mathematics that underpins their application to solve a range of engineering problems.

How you will be assessed

This unit is externally assessed by an unseen paper-based examination. The examination is set and marked by Pearson. Throughout this unit you will find practice activities that will help you to prepare for the examination. At the end of the unit you will also find help and advice on how to prepare for and approach the examination. The examination must be taken under examination conditions, so it is important that you are fully prepared and familiar with the application of the principles covered in the unit. You will also need to learn key formulae and be confident in carrying out calculations accurately. A scientific calculator and knowledge of how to use it effectively will be essential.

The examination will be two hours long and will contain a number of short- and long-answer questions. Assessment will focus on applying appropriate principles and techniques to solving problems. Questions may be focused on a particular area of study or require the combined use of principles from across the unit. An *Information Booklet of Formulae and Constants* will be available during the examination.

This table contains the skills that the examination will be designed to assess.

Assessment objectives	
AO1	Recall basic engineering principles and mathematical methods and formulae
AO2	Perform mathematical procedures to solve engineering problems
AO3	Demonstrate an understanding of electrical, electronic and mechanical principles to solve engineering problems
AO4	Analyse information and systems to solve engineering problems
AO5	Integrate and apply electrical, electronic and mechanical principles to develop an engineering solution

This table contains the areas of essential content that learners must be familiar with prior to assessment.

Essential content	
A1	Algebraic methods
A2	Trigonometric methods
B1	Static engineering systems
B2	Loaded components
C1	Dynamic engineering systems
D1	Fluid systems
D2	Thermodynamic systems
E1	Static and direct current electricity
E2	Direct current circuit theory
E3	Direct current networks
F1	Magnetism
G1	Single-phase alternating current theory

Getting started

To get started, have a quick look through each topic in this unit and then in small groups discuss why you think these areas are considered important enough to be studied by everyone taking a BTEC National Engineering course. Pick one or more topics and discuss how they might be relevant to a product, activity or industry you are familiar with.



A

Algebraic and trigonometric mathematical methods

Engineers have to be confident that the solutions they devise to address practical problems are based on sound scientific principles. In order to do this, they often have to solve complex mathematical problems, so they must be comfortable and competent when working with algebra and trigonometry.

A1 Algebraic methods

Algebra allows relationships between variables to be expressed in mathematical shorthand notation that can be manipulated to solve problems. In this part of the unit, we will consider algebraic **expressions** involving indices and logarithms.

Indices

Even if you do not yet recognise the term, you will already be familiar with the use of indices in common mathematical expressions. For example:

- ▶ 3×3 is otherwise known as 'three squared' or 3^2 in mathematical notation using indices.
- ▶ $5 \times 5 \times 5$ is otherwise known as 'five cubed' or 5^3 in mathematical notation using indices.

The two parts of the notation used to describe indices are called the base and the index. For example:

- ▶ In the expression 3^2 the **base** is 3 and the **index** is 2.

Often in engineering mathematics we have to consider situations where we do not yet know values for the numbers involved. We use algebra to represent unknown numbers with letters or symbols. When applied to indices:

- ▶ $a \times a = a^2$, where a is the base and 2 is the index.
- ▶ $b \times b \times b = b^3$, where b is the base and 3 is the index.

Where the index is also unknown, it can be represented by a letter as well, for example:

- ▶ a^n , where a is the base and n is the index.
- ▶ b^m , where b is the base and m is the index.

The laws of indices

- ▶ When dealing with **equations** that contain terms involving indices, there is a set of basic rules that you can apply to simplify and help solve them. These are the laws of indices. They are summarised in **Table 1.1**.

▶ **Table 1.1** The laws of indices

Operation	Rule
Multiplication	$a^m \times a^n = a^{m+n}$
Division	$\frac{a^m}{a^n} = a^{m-n}$
Powers	$(a^m)^n = a^{m \times n}$
Reciprocals	$\frac{1}{a^n} = a^{-n}$
Index = 0	$a^0 = 1$
Index = $\frac{1}{2}$ or 0.5	$a^{\frac{1}{2}} = \sqrt{a}$
Index = $\frac{1}{n}$	$a^{\frac{1}{n}} = \sqrt[n]{a}$
Index = 1	$a^1 = a$

Key terms

Expression – a mathematical statement such as $a^2 + 3$ or $3a - t$. Expressions can easily be recognised because they do not contain an equals sign.

Base – the term that is raised to an index, power or exponent. For example, in the expression 4^3 the base is 4.

Index – the term to which the base is raised. For example, in the expression 4^3 the index is 3. The index may also be called the power or exponent. The plural of 'index' is 'indices'.

Equation – used to equate two expressions that have equal value, such as $a^2 + 3 = 19$ or $t - 1 = 3a + 12$. Equations can easily be recognised because they always contain an equals sign.

Worked Example

You can use the laws of indices to simplify expressions containing indices.

- $a^2 a^4 a^0 a^{-3.2} = a^{2+4+0-3.2} = a^{2.8}$
- $(\sqrt{a})^3 a^{-1} = (a^{0.5})^3 a^{-1} = a^{1.5} a^{-1} = a^{1.5-1} = a^{0.5}$
or \sqrt{a}
- $\frac{a^{\frac{3}{2}} b^2 a^{-1}}{b} = a^{1.5} b^2 a^{-1} b^{-1} = a^{1.5-1} b^{2-1} = a^{0.5} b^1$
or $b\sqrt{a}$
- $\frac{a^{-1} b^{-1} a^{\frac{1}{2}}}{b^{-2}} = a^{-1} b^{-1} a^{0.5} b^2 = a^{-1+0.5} b^{-1+2} = a^{-0.5} b^1$
or $\frac{b}{\sqrt{a}}$
- $a^{-3}(b^{-2})^2 a^{3.5} = a^{-3+3.5} b^{-4} = a^{0.5} b^{-4}$ or $\frac{\sqrt{a}}{b^4}$

Logarithms

Logarithms are very closely related to indices.

The **logarithm** of a number (N) is the power (x) to which a given base (a) must be raised to give that number.

- ▶ In general terms, where $N = a^x$, then $\log_a N = x$.

In engineering, we encounter mainly **common logarithms**, which use base 10, and **natural logarithms**, which use base e .

The Euler number (e) is a mathematical constant that approximates to 2.718. You will come across this again later in the section dealing with natural exponential functions.

Common logarithms

Common logarithms are logarithms with base 10.

- ▶ Where $N = 10^x$, then $\log_{10} N = x$.

- ▶ When using common logarithms, there is no need to include the base 10 in the notation, so where $N = 10^x$, you can write simply $\log N = x$.

This corresponds to the log function on your calculator.

Natural logarithms

In a similar way, when dealing with natural logarithms with base e :

- ▶ Where $N = e^x$, then $\log_e N = x$.
- ▶ Natural logarithms with base e are so important in mathematics that they have their own special notation, where $\log_e N$ is written as $\ln N$. So where $N = e^x$, then $\ln N = x$.

This corresponds to the ln function on your calculator.

The laws of logarithms

There are a number of standard rules that can be used to simplify and solve equations involving logarithms. These are the laws of logarithms. They are summarised in **Table 1.2**.

▶ **Table 1.2** The laws of logarithms

Operation	Common logarithms	Natural logarithms
Multiplication	$\log AB = \log A + \log B$	$\ln AB = \ln A + \ln B$
Division	$\log \frac{A}{B} = \log A - \log B$	$\ln \frac{A}{B} = \ln A - \ln B$
Powers	$\log A^n = n \log A$	$\ln A^n = n \ln A$
Logarithm of 0	$\log 0 = \text{not defined}$	$\ln 0 = \text{not defined}$
Logarithm of 1	$\log 1 = 0$	$\ln 1 = 0$



PAUSE POINT

Use your calculator to practise finding the common and natural logarithms of a range of values. Be sure to include whole numbers and decimal fractions less than 1.

What happens when you try to find the logarithm of 0 or a negative number using your calculator?

Hint

From the relationship between logarithms and indices you know that if $\log 0 = x$, then $0 = 10^x$. What value must x take if $10^x = 0$?

Extend

Take a few moments to think about why $\log 0$ and $\log -1$ are not defined and will produce an error on your calculator.

Worked Examples

- 1 Use the laws of logarithms to solve the equation $\log 14^{(1-x)} = \log 9^{(x+3)}$.

Solution

$$\begin{aligned}\log 14^{(1-x)} &= \log 9^{(x+3)} \\ (1-x)\log 14 &= (x+3)\log 9 \\ \log 14 - x\log 14 &= x\log 9 + 3\log 9 \\ \log 14 - 3\log 9 &= x\log 9 + x\log 14 \\ \log 14 - 3\log 9 &= x(\log 9 + \log 14) \\ x &= \frac{\log 14 - 3\log 9}{\log 9 + \log 14} = -0.817 \text{ (rounded} \\ &\quad \text{to 3 significant figures (s.f.))}\end{aligned}$$

Always check your solution by substituting the unrounded value of the solution back into the original equation:

$$\begin{aligned}\log 14^{(1-x)} &= \log 14^{[1 - (-0.817\dots)]} = 2.08 \text{ (to 3 s.f.)} \\ \log 9^{(x+3)} &= \log 9^{(-0.817\dots + 3)} = 2.08 \text{ (to 3 s.f.)}\end{aligned}$$

- 2 Use the laws of logarithms to solve the equation $\ln 3t = 2 \ln \frac{12}{t} + 2$.

Solution

$$\begin{aligned}\ln 3t &= 2 \ln \frac{12}{t} + 2 \\ \ln 3 + \ln t &= 2(\ln 12 - \ln t) + 2 \\ \ln 3 + \ln t &= 2 \ln 12 - 2 \ln t + 2 \\ 3 \ln t &= 2 \ln 12 - \ln 3 + 2 \\ \ln t &= \frac{2 \ln 12 - \ln 3 + 2}{3} = 1.957\dots\end{aligned}$$

You can now use the general relationship that where $\ln N = x$, then $N = e^x$.

In this case, $\ln t = 1.957\dots$, so $t = e^{1.957\dots}$, which you can use your calculator to evaluate.

$$t = 7.08 \text{ (to 3 s.f.)}$$

Always check your solution by substituting the unrounded value of the solution back into the original equation:

$$\begin{aligned}\ln 3t &= \ln 21.235\dots = 3.056 \text{ (to 4 s.f.)} \\ 2 \ln \frac{12}{t} + 2 &= 2 \ln 1.695\dots + 2 = 2 \times 0.5278\dots + 2 \\ &= 3.056 \text{ (to 4 s.f.)}\end{aligned}$$

Exponential growth and decay

Exponential functions where x and y are variables and N is a constant take the general form $y = N^x$.

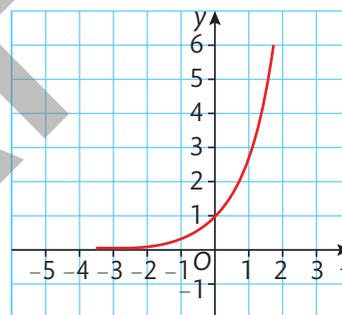
Some common systems found in engineering and in nature – such as charging capacitors, radioactive decay and light penetration in oceans – can be defined by exponential functions.

Several important growth and decay processes in engineering are defined by a special type of exponential function that uses the Euler number (e) as its base (N).

This is known as the natural exponential function and takes the basic form $y = e^x$.

If you plot the function $y = e^x$ as a graph (see **Figure 1.1**), then the curve it describes has two special characteristics that other exponential functions do not have.

- ▶ At any point on the curve the slope or gradient of the graph is equal to e^x .
- ▶ At $x = 0$ (where the curve intersects the y -axis) the graph has a slope or gradient of exactly 1.



▶ **Figure 1.1** Graph of the natural exponential function $y = e^x$

Examples of engineering formulae containing natural exponential functions are shown in **Table 1.3** for charging/discharging a capacitor.

▶ **Table 1.3** Example engineering formulae containing natural exponential functions

Description	Formula
Capacitor charge voltage	$V = V_0(1 - e^{-t/RC})$
Capacitor discharge voltage	$V = V_0 e^{-t/RC}$

Key term

Tangent – a straight line with a slope equal to that of a curve at the point where they touch.

II PAUSE POINT

Examine the characteristics of exponential functions by plotting a graph of $y = 2^x$ for values of x between 1 and 10.

Hint

Tabulate values of x and y . Ensure that your axes are of appropriate size and scale.

Extend

The gradient of the **tangent** at any point on the curve reflects the rate of change in y with respect to x at that point. What happens to this rate of change as x increases?



PAUSE POINT

Examine the characteristics of the natural exponential function by plotting a graph of $y = e^x$ for values of x between 1 and 10.

Extend

The gradient of the tangent at any point on the curve reflects the rate of change in y with respect to x at that point. Use your graph to demonstrate that for $y = e^x$ the tangent at any point has gradient e^x .

Problems involving exponential growth and decay

Worked Example – Exponential growth

When a capacitor with capacitance C is charged through a resistance R towards a final potential V_0 , the equation giving the voltage V across the capacitor at any time t is $V = V_0(1 - e^{-t/RC})$.

Given the values $C = 100 \mu\text{F}$, $R = 10 \text{ k}\Omega$ and $V_0 = 3 \text{ V}$, calculate V when $t = 3 \text{ s}$.



▶ A 100 μF capacitor

Solution

Substitute the given values into the formula, remembering to convert all values into appropriate units.

$$C = 100 \mu\text{F} = 100 \times 10^{-6} \text{ F}$$

$$R = 10 \text{ k}\Omega = 10 \times 10^3 \Omega$$

$$V_0 = 3 \text{ V}$$

$$t = 3 \text{ s}$$

Substituting these values into $V = V_0(1 - e^{-t/RC})$ gives $V = 3(1 - e^{-3})$.

So $V = 3(1 - 0.049\dots) = 2.85 \text{ V}$ (to 3 s.f.)

Solution

Substitute the given values into the formula, remembering to convert all values into appropriate units.

$$C = 47 \mu\text{F} = 47 \times 10^{-6} \text{ F}$$

$$R = 30 \text{ k}\Omega = 30 \times 10^3 \Omega$$

$$V_0 = 4 \text{ V}$$

$$V = 3.5 \text{ V}$$

Rearranging the formula $V = V_0 e^{-t/RC}$ to make t the subject:

$$e^{-t/RC} = \frac{V}{V_0}$$

$$-\frac{t}{RC} = \ln \frac{V}{V_0}$$

$$t = -RC \ln \frac{V}{V_0}$$

Substituting in the values gives:

$$t = -[30 \times 10^3 \times 47 \times 10^{-6}] \ln \frac{3.5}{4} = -1.41 \times -0.133\dots = 0.188 \text{ s (to 3 s.f.)}$$

Worked Example – Exponential decay

When a capacitor with capacitance C is discharged through a resistance R from an initial potential V_0 , the voltage V across the capacitor at any time t is given by the equation $V = V_0 e^{-t/RC}$.

Given the values $C = 47 \mu\text{F}$, $R = 30 \text{ k}\Omega$ and $V_0 = 4 \text{ V}$,

calculate the time t at which $V = 3.5 \text{ V}$.



▶ A 47 μF capacitor

Linear equations and straight-line graphs

In engineering, many simple systems behave in a linear fashion. **Table 1.4** shows some examples. When plotted graphically, linear relationships are characterised by a straight line with a constant gradient.

▶ **Table 1.4** Examples of engineering formulae describing linear relationships

Description	Formula
Standard general form	$y = mx + c$
Ohm's law	$V = IR$
Gravitational potential energy	$PE = mgh$
Electrical power	$P = IV$
Resistivity	$\rho = \frac{RA}{l}$
Linear motion	$v = u + at$

Linear equations

Linear equations contain unknowns that are raised to their first power only, such as x raised to its first power or x^1 (which is usually written simply as x). Linear equations take the general form $y = mx + c$.

Linear equations containing a single unknown quantity can be solved by rearranging to make the unknown quantity the **subject** of the equation.

Worked Example – Rearranging a linear equation

Rearrange $4x + 3 = 12$ to make x the subject of the equation.

Solution

$4x = 12 - 3 = 9$ Subtract 3 from both sides of the equation.

$x = \frac{9}{4}$ Divide both sides of the equation by 4.

If the same unknown quantity occurs more than once in a linear equation, then all the terms containing the unknown quantity need to be isolated on one side of the equation with all the other terms on the other side. This is called 'gathering like terms'.

Worked Example – Gathering like terms in a linear equation

Rearrange $9b + 6 = 14 - 2b$ to make b the subject of the equation.

Solution

$9b + 2b = 14 - 6$ Add $2b$ and subtract 6 on both sides of the equation.

$11b = 8$ Combine like terms.

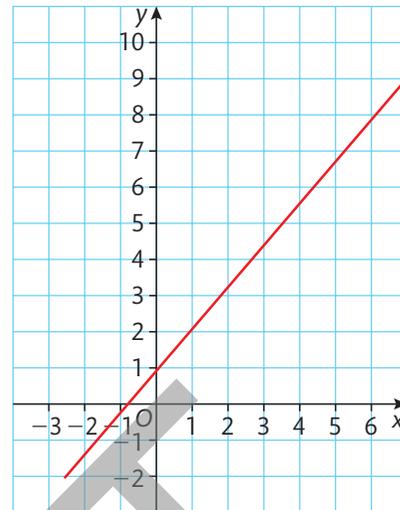
$b = \frac{8}{11}$ Divide both sides of the equation by 11.

Straight-line graphs

Graphs in mathematics usually use a horizontal x -axis and a vertical y -axis (known as Cartesian axes).

A linear equation with two unknowns can be represented graphically by a straight line as shown in **Figure 1.2**.

In the general formula used to describe a linear equation, $y = mx + c$, m is the gradient of the line and c is the value of y where the line intercepts the y -axis (when $x = 0$).



► **Figure 1.2** The graphical representation of a linear equation is a straight line

The **gradient** of a linear equation can be:

- positive – the y values increase linearly as the x values increase (the line slopes up from left to right)
- negative – the y values decrease linearly as the x values increase (the line slopes down from left to right)
- zero – the y values stay the same as the x values increase (the line is parallel to the x -axis).

Key terms

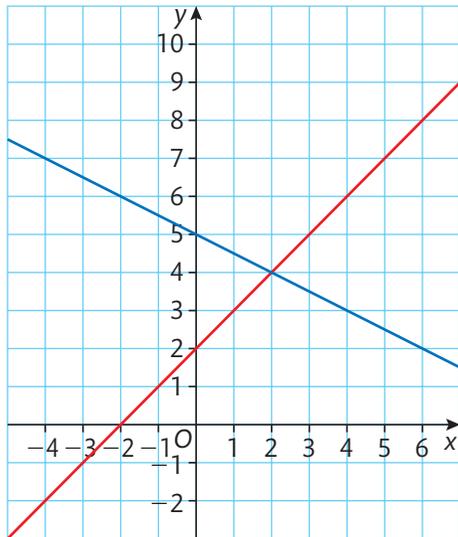
Subject of an equation – a single term becomes the subject of an equation when it is isolated on one side of the equation with all the other terms on the other side. For example, in the equation $y = 4x + 3$, the y term is the subject.

Gradient – also called 'slope', measures how steep a line is. It is calculated by picking two points on the line and dividing the change in height by the change in horizontal distance, or $\frac{\text{change in } y \text{ value}}{\text{change in } x \text{ value}}$.

Solving pairs of simultaneous linear equations

Sometimes in engineering it is necessary to solve systems that involve pairs of independent equations that share two unknown quantities. There are several examples of this in the assessment activity practice questions at the end of this section.

When you solve simultaneous equations, you are determining values for the unknowns that satisfy both equations. This is easiest to see by considering two independent linear equations in x and y that are plotted graphically on the same axes (see **Figure 1.3**). The only position where the same values of x and y satisfy both equations is where the lines intersect. You can read these values from the x -axis and the y -axis. In this example, the solution is $x = 2$ and $y = 4$.



► **Figure 1.3** The solution of the simultaneous linear equations $y = x + 2$ and $2y = -x + 10$ is given by the intersection point of the two lines

Of course you could always plot graphs of the linear equations when solving simultaneous equations, but this is a time-consuming approach and less accurate than the alternative algebraic methods.

The two main algebraic methods – **substitution** and **elimination** – are best explained by working through examples.

Worked Example – Substitution method

Solve this pair of simultaneous equations:

$$y = x + 2 \quad (1)$$

$$2y = -x + 10 \quad (2)$$

Solution

Substitute equation (1) into equation (2) to get $2(x + 2) = -x + 10$.

So $2x + 4 = -x + 10$ Multiply out the brackets.

$$2x + x = 10 - 4 \quad \text{Collect like terms.}$$

$$3x = 6$$

$$x = 2 \quad \text{Divide both sides by 3.}$$

Now substitute $x = 2$ into equation (1) in order to find y : $y = 2 + 2 = 4$.

To check your solution, substitute $x = 2$ into equation (2) and see if you get the same value for y :

$$2y = -2 + 10 = 8$$

$$y = 4$$

Worked Example – Elimination method

Solve the same pair of simultaneous equations as before:

$$y = x + 2 \quad (1)$$

$$2y = -x + 10 \quad (2)$$

Solution

Multiply equation (1) by 2 so that the y terms in both equations become the same:

$$2y = 2x + 4 \quad (3)$$

Subtract equation (2) from equation (3) to eliminate the identical terms in y :

$$(2x + 4) - (-x + 10) = 0$$

$$3x - 6 = 0$$

$$3x = 6$$

$$x = 2$$

As before, substitute $x = 2$ into equation (1) to find $y = 4$, and then check your solution in equation (2).

Coordinates on Cartesian grid

Any point on a plotted linear equation can be expressed by its **coordinates** (x, y) .

In the example shown in **Figure 1.3**, the two lines cross at the point where $x = 2$ and $y = 4$. Its coordinates are $(2, 4)$.

Any point on the x -axis has a y -coordinate of zero. For example, the point $(4, 0)$ lies on the x -axis.

Similarly, any point on the y -axis has an x -coordinate of zero, so the point $(0, 4)$ lies on the y -axis.

The point $(0, 0)$, where both x - and y -coordinates are zero, is called the **origin** and is represented by 'O'.

Quadratic equations

Quadratic equations contain unknowns that are raised to their second power, such as x^2 . They take the general form $y = ax^2 + bx + c$. In this equation x and y are unknown variables, and coefficients a , b and c are constants that will be discussed later.

A quadratic equation with two unknowns can be represented graphically by a curve. An example is shown in **Figure 1.4**.

II PAUSE POINT

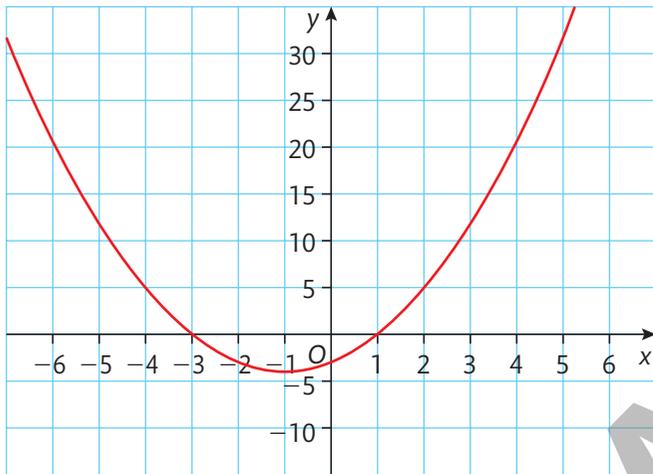
Show the efficiency of using an analytical approach by solving the simultaneous equations in the above worked examples with the graphical method.

Hint

Tabulate values of x and y for each equation. Ensure that the scales of your axes are appropriate.

Extend

Compare and evaluate the use of graphical and analytical methods in solving simultaneous equations. Are there any situations where a graphical approach might be preferred?



► **Figure 1.4** Graph of the quadratic equation $y = x^2 + 2x - 3$

You will often have to solve quadratic equations in engineering problems. This can be done by arranging the equation in the general form for quadratics and finding the two values of x for which $y = 0$. These are called the **roots** of the equation and are shown on the graph as the points where the line intersects the x -axis.

Two algebraic methods of solving quadratic equations are **factorisation** and using the **quadratic formula**. Before you consider factorising, it is a good idea to remind yourself how to multiply an expression by a number or by a symbol or more complicated expression.

Worked Example – Multiplying an expression by a number

Multiply each term of the expression by the number:

- $4(3x - 2) = 12x - 8$
- $5\left(\frac{x}{3} - 2\right) = \frac{5x}{3} - 10$.

Worked Example – Multiplying an expression by a symbol or another expression

Multiply out the brackets term by term, then collect like terms:

- $2x(3x - 2) = 6x^2 - 4x$
- $(x + 1)(x - 2) = x^2 - 2x + x - 2 = x^2 - x - 2$
- $(x + 2)(3x - 2) = 3x^2 - 2x + 6x - 4 = 3x^2 + 4x - 4$.

Factorisation

You can think of factorisation as the opposite of multiplying out, although it is generally somewhat more difficult. Factorising can be done in several ways, using some or all of the following techniques.

► Extracting common factors

Common factors are terms by which some or all parts of an expression can be divided. These can be extracted as follows:

- $ax + ay = a(x + y)$ where a is the common factor
- $2x + 4y = 2(x + 2y)$ where 2 is the common factor
- $a(x + 2) + b(x + 2) = (x + 2)(a + b)$ where $(x + 2)$ is the common factor.

► Grouping

When two variables x and y are present in the same expression, they must be grouped together before factorising. For example:

$$\begin{aligned} 32x^2 + 18y^3 + 8x + 9y^2 &= 18y^3 + 9y^2 + 32x^2 + 8x \\ &= 9y^2(2y + 1) + 8x(4x + 1) \end{aligned}$$

Sometimes this might mean that you have to multiply out the expression first. For example:

$$\begin{aligned} 4(2y^2 - x^2) + y(y^2 + 7) &= 8y^2 - 4x^2 + y^3 + 7y \\ &= y^3 + 8y^2 + 7y - 4x^2 \\ &= y(y^2 + 8y + 7) - 4x^2 \end{aligned}$$

The second example can be factorised further by using the methods for factorising quadratics described in the next section.

Solving quadratic equations using factorisation

Let us take an example of a quadratic equation that you might need to solve as part of an engineering problem:

$$3x + 8 = 5x + x^2$$

- ▶ First, rearrange the equation (if necessary) to make one side of the equation zero.

$$0 = 5x + x^2 - 3x - 8 \quad \text{Move all the terms to one side of the equation.}$$

$$0 = x^2 + 2x - 8 \quad \text{Arrange into the standard form for quadratic equations.}$$

- ▶ The next step is to factorise the right-hand side of this equation.

Key term

Coefficient – a number or symbol that multiplies a variable. For example, in the expression $3x$, 3 is the coefficient of the variable x .

To do this, you need to find two expressions that when multiplied together give $x^2 + 2x - 8$. This looks difficult, but there are a few general guidelines that will help:

- ▶ If the **coefficient** of the x^2 term (a) is 1, then the coefficient of each of the x terms in the factors will also be 1.
- ▶ When the number terms in the two factors are multiplied together, the product must equal the number term (c) in the quadratic expression.
- ▶ If the coefficient of the x^2 term (a) is 1, then the coefficient of the x term (b) in the quadratic expression is equal to the sum of the number terms in the two factors.

Applying these guidelines to $x^2 + 2x - 8$, you know that:

- ▶ the coefficients of the x terms in the factors will be 1
- ▶ the product of the number terms in the factors will be -8
- ▶ the sum of the number terms in the factors will be 2.

Finding the terms to put in the brackets is then often a case of trying different values until you identify those that meet the required criteria:

$$x^2 + 2x - 8 = (x - 2)(x + 4)$$

- ▶ Equate each of the factors to zero to obtain the roots of the equation.

$$(x - 2) = 0, \text{ so } x = 2$$

$$(x + 4) = 0, \text{ so } x = -4$$

- ▶ Check that each root satisfies the original equation by substitution.

$$3x + 8 = 5x + x^2$$

$$\text{When } x = -4: -12 + 8 = -20 + 16$$

$$\text{or } -4 = -4$$

$$\text{When } x = 2: 6 + 8 = 10 + 4$$

$$\text{or } 14 = 14$$

- ▶ Check to ensure that each solution is reasonable in the context of the question and clearly state the solution.

There are two values of x for which the quadratic equation $3x + 8 = 5x + x^2$ is true. These are $x = 2$ and $x = -4$.

See the worked example on page 11.

Some examples of engineering formulae describing quadratic relationships are given in **Table 1.5**.

- ▶ **Table 1.5** Examples of engineering formulae describing quadratic relationships

Description	Formula
Standard general form	$ax^2 + bx + c = 0$
Displacement	$s = ut + \frac{1}{2}at^2$
Total surface area of a cylinder	$A = 2\pi r^2 + 2\pi rh$

Solving quadratic equations using the quadratic formula

Sometimes it is difficult to use factorisation to find the roots of a quadratic equation. In such cases, you can use an alternative method.

The formula for the roots of a quadratic equation arranged in the standard form is:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Worked Example

A train runs along a level track with a velocity of 5 m s^{-1} . The driver presses the accelerator, causing the train to increase speed by 2 m s^{-2} . The motion of the train is defined by the equation $s = ut + \frac{1}{2}at^2$

where: s is displacement (the distance travelled) (m)
 u is initial velocity (m s^{-1})
 t is time (s)
 a is acceleration (m s^{-2}).

Calculate the time the train takes to travel a distance of 6 m.



► Train running along a track

Solution

Rearrange the equation into the general form for a quadratic, making one side equal to zero:

$$\frac{1}{2}at^2 + ut - s = 0$$

Substitute the values $u = 5$, $s = 6$ and $a = 2$ into the equation:

$$\begin{aligned} \frac{1}{2}(2)t^2 + 5t - 6 &= 0 \\ t^2 + 5t - 6 &= 0 \end{aligned}$$

The left-hand side can be factorised by inspection to give $(t + 6)(t - 1) = 0$.

This equation is true when either of the factors is equal to zero, so $t = -6$ or $t = 1$.

It is important to check these solutions by substituting back into the original quadratic equation given in the question: $s = ut + \frac{1}{2}at^2 = 5t + t^2$.

You will find that when $t = 1$, $s = 6$ and when $t = -6$, $s = 6$.

This confirms that the solutions determined by finding the roots of the quadratic equation are mathematically correct, because they both give the distance required in the question.

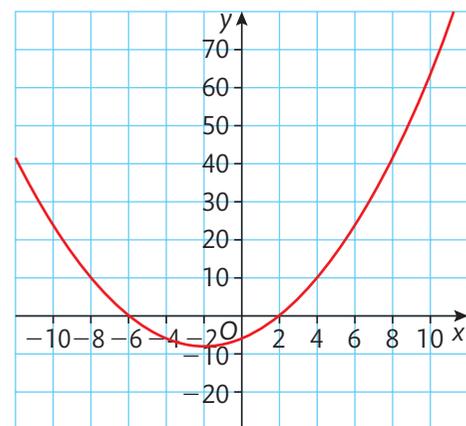
However, you must now consider whether both solutions actually fit the practical situation in the real world.

In this case $t = -6$ doesn't make sense as a solution, because it would mean that the train reached the required distance 6 seconds before it started accelerating.

The roots can be seen more clearly when illustrated graphically, as in **Figure 1.5**.

Finally, you should clearly state the solution to the problem:

The time taken for the train to travel a distance of 6 m is 1 s.



► **Figure 1.5** Graph showing roots of the quadratic expression $t^2 + 5t - 6$ at $t = -6$ and $t = 1$

Worked Example

A ball is thrown vertically upwards with an initial velocity of 16 m s^{-1} . The height reached by the ball is given by the equation $h = ut - \frac{1}{2}gt^2$

where: h is the vertical height (m)

u is the initial vertical velocity (m s^{-1})

t is time (s)

g is the gravitational field strength (9.81 m s^{-2})

Calculate the values of t at which the height of the ball is 6 m above the point where it was released.

Solution

Rearrange the equation into the general form for a quadratic equation, making one side equal to zero:

$$\frac{1}{2}gt^2 - ut + h = 0$$

Substitute the values $u = 16$, $h = 6$ and $g = 9.81$ into the equation:

$$\frac{1}{2} \times 9.81t^2 - 16t + 6 = 0$$

$$4.905t^2 - 16t + 6 = 0$$

It would be really difficult to find the roots of this equation by factorisation, so use the

general formula for solving quadratics: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

By comparing the formula with the standard form $ax^2 + bx + c$, you can see that

$$a = 4.905, b = -16, c = 6$$

$$\text{So } t = \frac{-(-16) \pm \sqrt{(-16)^2 - (4 \times 4.905 \times 6)}}{2 \times 4.905}$$

$$= \frac{16 \pm \sqrt{256 - 117.72}}{9.81}$$

$$= \frac{16 \pm 11.76}{9.81}$$

$$= 0.432 \text{ or } 2.83$$

It is important to check these solutions by substituting back into the original quadratic equation given in the question: $h = ut - \frac{1}{2}gt^2 = 16t - 4.905t^2$.

You will find that when $t = 0.432\dots$, $h = 6$ and when $t = 2.83\dots$, $h = 6$. Be careful to use the unrounded values when you do the checks.

This confirms that the solutions determined by finding the roots of the quadratic equation are correct because they both give the required height stated in the question.

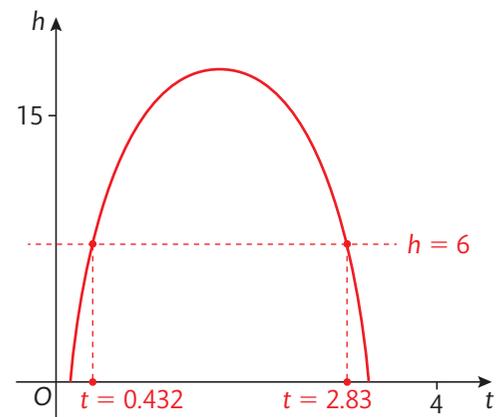
The next thing to do is consider whether these solutions actually fit the practical situation for which they were generated. In this case, it makes sense that as the ball moves up it will pass the height of 6 m. Gravity will slow this ascent until the ball actually stops momentarily before it falls back towards the ground. On its way down it will pass the height of 6 m once again.

This can be seen more clearly when illustrated graphically, as in

Figure 1.6.

Finally, you should clearly state the solution to the problem:

The values of t at which the height of the ball is 6 m above the point where it was released are 0.432 s and 2.83 s.



► **Figure 1.6** Graph of $h = 16t - 4.905t^2$, showing when it reaches the height of 6 m

II PAUSE POINT

Discuss with a colleague or as a group why methods of finding the roots of quadratic equations are important and useful mathematical tools for engineers.

Hint

Consider how else you might solve problems that involve quadratic equations.

Extend

Suppose that you want to solve the quadratic equation $y = x^2 + 4x + 6$ to find values of x when $y = 1$. This would mean finding the roots of the quadratic expression $x^2 + 4x + 5$. Can you solve this? Draw a graph of the function $y = x^2 + 4x + 5$ to help explain why not.

A2 Trigonometric methods

Angular measurement

You are already familiar with angular measurements made in **degrees**. In practical terms, this is the most common way to define an angle on an engineering drawing that a technician might use when manufacturing a component in the workshop.

However, there is another unit of angular measurement, called the **radian**, which is used extensively in engineering calculations.

Key terms

Degree (symbol: °) – one degree is $\frac{1}{360}$ th of a complete circle. A complete circle contains 360° .

Radian (symbol: rad or °) – one radian is the angle **subtended** at the centre of a circle by two radii of length r that describe an arc of the same length r on the circumference. A complete circle contains 2π rad.

Subtend – to form an angle between two lines at the point where they meet.

Circular measurement

One revolution of a full circle contains 360° or 2π radians.

It is reasonably straightforward to convert angles stated in degrees to radians and vice versa.

Given that 2π radians = 360°

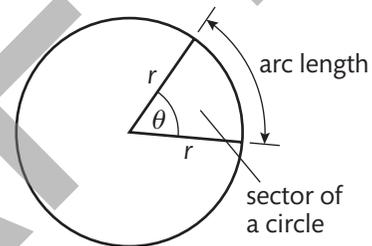
$$1 \text{ radian} = \frac{360^\circ}{2\pi} \approx 57.3^\circ \text{ (to 3 s.f.)}$$

$$\text{and } 1^\circ = \frac{2\pi}{360} \approx 0.0175 \text{ rad (to 3 s.f.)}$$

The use of radians makes it straightforward to calculate some basic elements of circles with the general formulae shown in **Table 1.6**, where the angle θ is measured in radians.

► **Table 1.6** General formulae for circular measurements (see **Figure 1.7**)

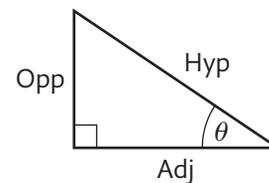
Arc length	$= r\theta$
Circumference of a circle	$= r(2\pi) = 2\pi r$
Area of a sector	$= \frac{1}{2}r^2\theta$
Area of a full circle	$= \frac{1}{2}r^2(2\pi) = \pi r^2$



► **Figure 1.7** Arc length and sector of a circle

Triangular measurement

In right-angled triangles we name the three sides in relation to the right angle and one of the other two angles, θ (see **Figure 1.8**).



► **Figure 1.8** Trigonometric naming conventions for a right-angled triangle

- The side opposite the right angle is the **hypotenuse** (hyp).
- The side next to the angle θ is the **adjacent** (adj) side.
- The side opposite the angle θ is the **opposite** (opp) side.

The ratios of the lengths of these sides are given specific names and are widely used in engineering (see **Figures 1.9–1.11**):

- **sine** (sin), where

$$\sin \theta = \frac{\text{opp}}{\text{hyp}}$$

► **cosine** (cos), where

$$\cos \theta = \frac{\text{adj}}{\text{hyp}}$$

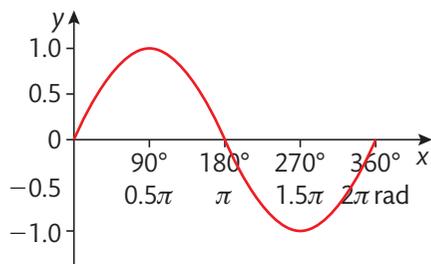
► **tangent** (tan), where

$$\tan \theta = \frac{\text{opp}}{\text{adj}}$$

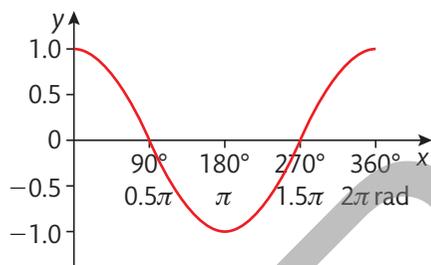
From these definitions it can also be deduced that

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

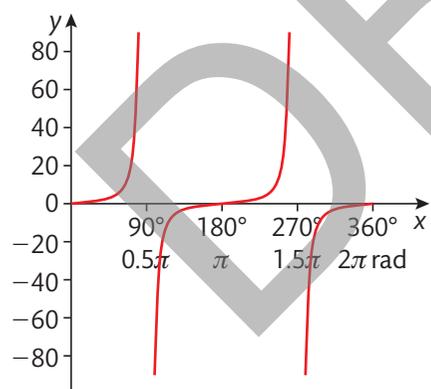
Graphs of the trigonometric functions



► **Figure 1.9** Graph of $y = \sin \theta$



► **Figure 1.10** Graph of $y = \cos \theta$



► **Figure 1.11** Graph of $y = \tan \theta$

When plotted graphically, both the sine and the cosine functions generate periodic waveforms. Both functions vary between a maximum of 1 and a minimum of -1 and so are said to have an **amplitude** of 1. Both functions have a **period** of 360° or 2π radians, after which the cycle repeats.

The tangent function does not generate a smooth waveform, although the graph is still periodic with a period of 180° or π radians.

Some values for the trigonometric ratios are given in **Table 1.7**.

► **Table 1.7** Values of the trigonometric ratios for angles between 0° and 360°

θ°	$\theta \text{ rad}$	$\sin \theta$	$\cos \theta$	$\tan \theta$
0	0	0	1	0
30	0.52	0.50	0.87	0.58
60	1.05	0.87	0.50	1.73
90	1.57	1.00	0	$\pm\infty$
120	2.09	0.87	-0.50	-1.73
150	2.62	0.50	-0.87	-0.58
180	π	0	-1.00	0
210	3.67	-0.50	-0.87	0.58
240	4.19	-0.87	-0.50	1.73
270	4.71	-1.00	0	$\pm\infty$
300	5.24	-0.87	0.50	-1.73
330	5.76	-0.50	0.87	-0.58
360	2π	0	1.00	0

Sine and cosine rules

The basic definitions of the trigonometric functions sine, cosine and tangent only apply to right-angled triangles. However, the sine and cosine rules can be applied to any triangle of the form shown in **Figure 1.12**.

► The sine rule:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

II PAUSE POINT

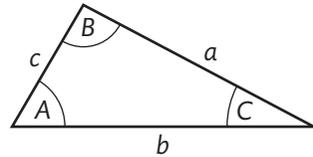
Check that you can set up and use a scientific calculator in both degree and radian modes.

Hint

Look for the set-up screen on your calculator and select Rad. You will find that the D (for degrees) usually displayed at the top of the screen changes to R (for radians).

Extend

Before carrying out any work involving trigonometry, you should check that you are using the appropriate setting on your calculator. Use some simple examples to show what might happen if you get it wrong.



► **Figure 1.12** Naming conventions when applying the sine rule and the cosine rule

- The cosine rule can take three different forms depending on the missing value to be determined:

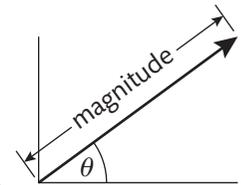
$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

Vectors and their applications

Many quantities encountered in engineering, such as force and velocity, are only fully described when magnitude, direction and sense are known (see **Figure 1.13**). Such quantities are called **vectors**. When adding or subtracting vectors you must always take into account the direction in which they act.



► **Figure 1.13** A vector has magnitude, direction and sense

Diagrammatic representation of vectors

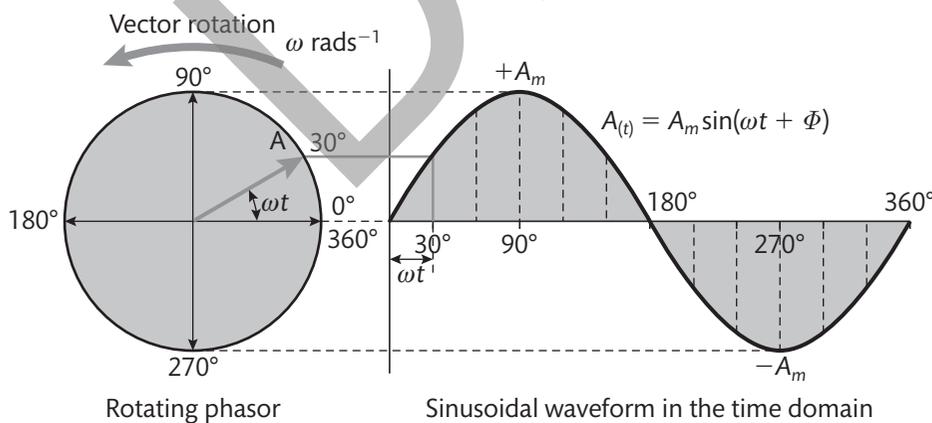
- The length of the arrow represents the magnitude of the vector.
- The angle θ specifies the direction of the vector.
- The head of the arrow specifies the positive sense of the vector.

Vector addition

To find the sum (or resultant) of two vectors v_1 and v_2 , you can represent the situation graphically by drawing a vector diagram. In **Figure 1.14** the two vectors are drawn to scale, forming a triangle or parallelogram from which the characteristics of the resultant vector (v_T) can be measured.

Phasors

Phasors are rotating vectors that are useful in analysing sinusoidal (sine-shaped) waveforms. **Figure 1.15** shows the relationship between a phasor and the sine wave it represents.

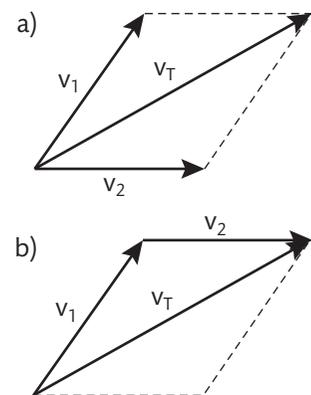


► **Figure 1.15** How the rotating phasor relates to the sinusoidal waveform

When using phasors in the analysis of alternating current, the length of the phasor represents the peak voltage (V) or amplitude of the sinusoidal waveform, and the phasor rotates about a point of origin with an **angular velocity** of ω .

Link

Graphical and analytical techniques for vector addition are discussed further in the section **B1 Static engineering systems**.



► **Figure 1.14** Vector addition can be done using a) the parallelogram rule or b) the triangle rule

At any fixed instant in time (t), the phasor will have a phase angle (θ) and its vertical component will be equal to the instantaneous voltage (v) on the corresponding sine wave.

So the instantaneous voltage (v) at any point on the waveform is

$$v = V \sin \theta$$

In terms of angular velocity, this gives

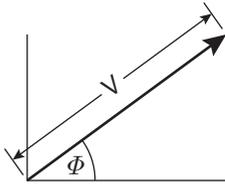
$$v = V \sin(\omega t)$$

This relationship is true when the waveform begins its cycle when $t = 0$. However, it is common to have a waveform that is said to lead or lag the standard waveform. This **phase difference** (see **Figure 1.16**) is expressed as an angle (Φ), so that

$$v = V \sin(\omega t + \Phi)$$

The angular velocity of the phasor (ω) is related to the **frequency** of the waveform (f) by

$$\omega = 2\pi f$$



► **Figure 1.16** Phasor representing the sinusoidal waveform $v = V \sin(\omega t + \Phi)$, where Φ is the phase shift from the standard waveform

II PAUSE POINT

Explain the difference between a vector and a phasor.

Hint

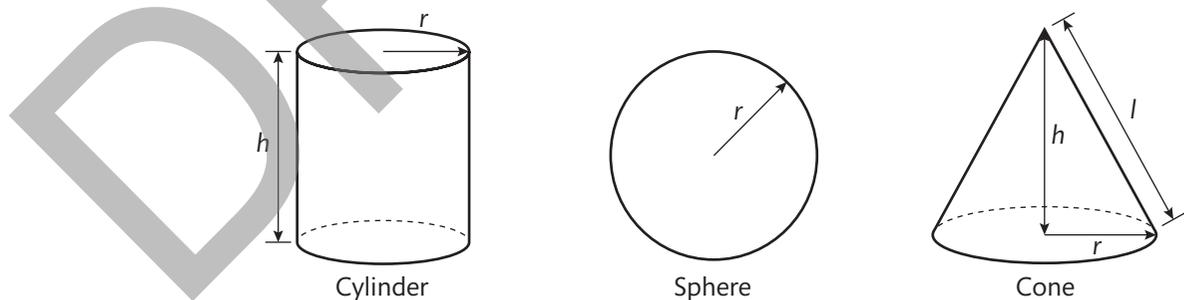
Consider the different types of systems they are used to describe.

Extend

Use a diagram to explain the relationship between a phasor and the sinusoidal waveform it describes.

Mensuration

It is very common for an engineer to need to calculate the surface area or volume of three-dimensional shapes (see **Figure 1.17**); for example, to determine the number of tiles required to line a swimming pool or to find the capacity of a cylindrical storage tank. There are several important formulae that can help you do this (see **Table 1.8**).



► **Figure 1.17** Some commonly encountered regular solids and their dimensions

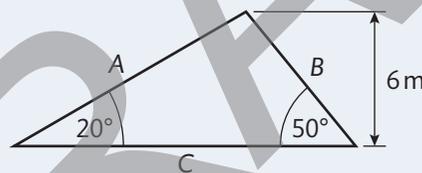
► **Table 1.8** Standard formulae for the surface area and volume of some regular solids

Regular solid	Curved surface area (CSA)	Total surface area (TSA)	Volume
Cylinder	$2\pi rh$	$2\pi rh + 2\pi r^2 = 2\pi r(h + r)$	$\pi^2 rh$
Cone	πrl	$\pi r^2 + \pi rl = \pi r(r + l)$	$\frac{1}{3}\pi r^2 h$
Sphere	$4\pi r^2$		$\frac{4}{3}\pi r^3$

Often, an apparently complex object can be broken down into a series of regular solids.

Assessment practice 1.1

- 1 A small boat is capable of a maximum velocity in still water of v km h⁻¹. On a journey upriver against a current with velocity v_c km h⁻¹, the boat travels at its full speed for 2 hours and covers 16 km. On a journey back downriver with the same current flow, the boat travels at full speed for 1 hour and 20 minutes and covers 18 km. Establish and then solve a pair of simultaneous linear equations to determine the maximum velocity of the boat v and the current v_c . (4 marks)
- 2 Scrap transformers have been collected for recycling. In total, 203 transformers weigh 403.4 kg. It is found that the transformers come in two different types, which weigh 1.8 kg and 2.3 kg, respectively. Establish and then solve a pair of simultaneous linear equations to determine the number of each type of transformer collected. (4 marks)
- 3 The motion of two vehicles is described by the linear equations
- $$27 = v - 3t$$
- $$13 = v - 4t$$
- By solving this pair of simultaneous equations determine:
- the time t at which both vehicles will be travelling with the same velocity
 - the corresponding velocity v at that point. (2 marks)
- 4 The time t (years) taken for a radioactive isotope contained in stored nuclear waste to decay to 5% of its original quantity is given by the equation
- $$5 = 100 \times 2^{-\frac{t}{1622}}$$
- Solve the equation to find the time t .
Show evidence of the use of the laws of logarithms in your answer. (2 marks)
- 5 An engineer has been given a drawing (see **Figure 1.18**) of a triangular plate with minimal dimensional detail.

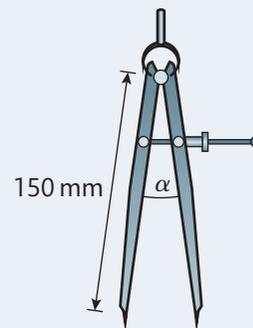


► **Figure 1.18** Engineering drawing of a triangular plate

Calculate the lengths of the sides A , B and C .

(4 marks)

- 6 A pair of dividers used for marking out has legs that are 150 mm long, as shown in **Figure 1.19**. The angle α between the legs can be set to a maximum of 60° by rotating the adjustment screw.



► **Figure 1.19** A pair of dividers

(4 marks)

- Use the sine rule to calculate the maximum distance that can be set between the points of the dividers when $\alpha = 60^\circ$.

- Use the cosine rule to calculate the value of angle α when the points of the dividers are set to a distance of 38 mm.

- 7 The displacement s of an accelerating body with respect to time t is described by the equation

$$s = t^2 - 7t + 12$$

Using factorisation, solve this equation to determine the values of t for which $s = 2$.

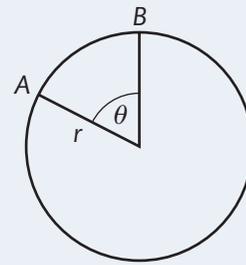
(3 marks)

- 8 Use the formula for solving quadratic equations to determine the radius r of an enclosed cylinder which has a total surface area (TSA) of 12.25 m² and a height h of 1.2 m, where

$$TSA = 2\pi rh + 2\pi r^2$$

(4 marks)

- 9 A technician has been given an engineering drawing of a circular component (see **Figure 1.20**) where the arc length AB is 23 cm and the radius r is 16 cm. Calculate the angle θ in degrees to allow the technician to use a Vernier protractor to mark out the component accurately. (2 marks)



► **Figure 1.20** Engineering drawing of a circular component

For each of the problems in this assessment practice, use the following stages to guide your progress through the task.

Plan

- What important information is provided in the question? Can this information be summarised or collated to make it easier to digest?
- Will a sketch help me to visualise what is being asked?

Do

- I am confident that I have interpreted the question correctly, and I have a plan to approach finding a solution.
- I recognise when a solution is clearly incorrect and will challenge my previous approach and try an alternative.

Review

- I can explain the engineering principles that underpin my approach to a task.
- I can clearly follow the method I used to complete a task and explain my reasoning at each stage.

B Static engineering systems

B1 Static engineering systems

Concurrent and non-concurrent coplanar forces

Mechanical systems often contain components that exert forces that push or pull other components. A force will tend to cause a change in the motion of an object, but in a static system all the forces are balanced, and so the system will either remain at rest or be in motion with constant velocity.

As mentioned in the previous section on mathematical methods, forces are vector quantities, so to define them fully we need to know their **magnitude**, **direction** and **sense**.

Where systems contain multiple forces, it is possible to represent the forces visually in a space diagram, free body diagram or vector diagram.

Consider a mass suspended by two wires under tension.

A **space diagram** is a sketch of the physical arrangement of the system being considered, like that shown in **Figure 1.21**.

Key terms

Concurrent forces – forces that all pass through a common point.

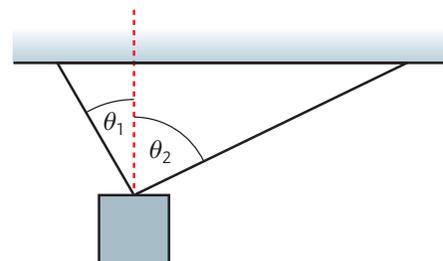
Non-concurrent forces – forces that do not all pass through the same point.

Coplanar forces – forces acting in the same two-dimensional plane.

Magnitude – the size of a force.

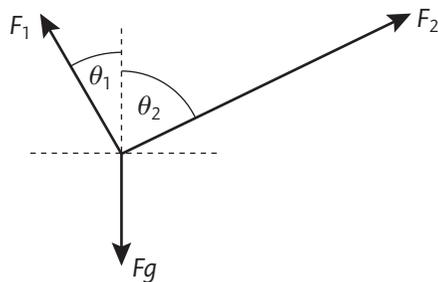
Direction – the orientation of the line in which the force is acting (the line of action).

Sense – the direction along the line of action in which the force acts.



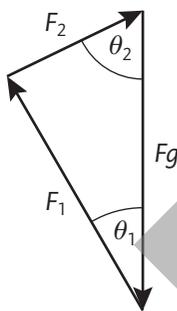
► **Figure 1.21** Space diagram showing an object suspended by two wires

A **free body diagram** is a sketch containing just the forces acting on the system or the part of the system that you are interested in, as shown in **Figure 1.22**.



► **Figure 1.22** Free body diagram showing the forces acting on the suspended mass (which is represented by a point)

A **vector diagram** is a sketch in which the lengths of the lines representing the force vectors correspond to the magnitudes of the respective forces. The vectors may also have been rearranged to form a triangle of forces as shown in **Figure 1.23**. (Similarly, larger numbers of force vectors can be arranged into a polygon of forces.)



► **Figure 1.23** Vector diagram with the three forces rearranged into a triangle

Resolution of forces in perpendicular directions

When analysing a system in which multiple forces act on a body, it may not be obvious what the overall effect of those forces will be. What we need is a method of adding the forces together to find their combined effect, known as the **resultant** force acting on the body. We can combine forces graphically by drawing a vector diagram, but here we will consider an alternative approach using the resolution of forces.

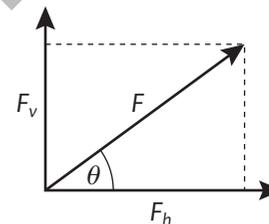
Key term

Resultant – the force that represents the combined effect of all the forces in a system.

One way to add or subtract vector quantities is to split each vector into components acting in specific, perpendicular directions. A force acting in any direction can be resolved into a vertical and a horizontal component. The horizontal components of multiple forces can be simply added to calculate a single horizontal force. Similarly, the vertical components of multiple forces can be simply added to calculate a single vertical force. These can then be recombined into a single resultant force that represents the combined effect of all the original individual forces.

Splitting any vector into its horizontal and vertical components will involve the use of trigonometry and/or Pythagoras' theorem.

Figure 1.24 shows how a single force F can be resolved into a horizontal component F_h and a vertical component F_v .



► **Figure 1.24** Force vector resolved into vertical and horizontal components

In the arrangement shown in **Figure 1.24**:

$$F_v = F \sin \theta$$

$$F_h = F \cos \theta$$

$$\frac{F_v}{F_h} = \tan \theta$$

$$F^2 = F_v^2 + F_h^2$$

When several forces act on a body, apply the same principles to each of the forces.



PAUSE POINT

Explain the difference between a space diagram, a free body diagram and a vector diagram.

Hint

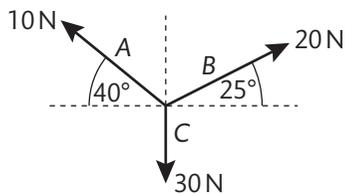
Draw an example of each for the same system.

Extend

How will the diagrams differ between concurrent and non-concurrent systems of forces?

Worked Example

Find the resultant and **equilibrant** for the system of concurrent forces shown in **Figure 1.25**.



► **Figure 1.25** System of three concurrent coplanar forces

Solution

First, establish the convention that the positive vertical direction is up and the positive horizontal direction is to the right.

Resolve each vector into its vertical and horizontal components using the formulae on the previous page.

Find the sum of the vertical components (the Greek letter Σ is shorthand notation for 'sum'):

$$\begin{aligned}\Sigma F_v &= A_v + B_v + C_v = 10 \sin 40 + 20 \sin 25 - 30 \\ &= -15.12 \text{ N}\end{aligned}$$

Find the sum of the horizontal components:

$$\begin{aligned}\Sigma F_h &= A_h + B_h + C_h = -10 \cos 40 + 20 \cos 25 + 0 \\ &= +10.47 \text{ N}\end{aligned}$$

Figure 1.26 shows that the resultant force F_r is acting in the positive sense in a direction θ° below the horizontal, where:

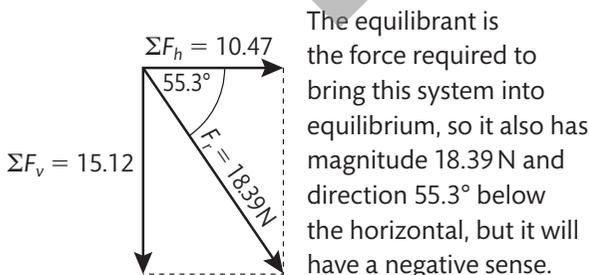
$$\tan \theta = \frac{15.12}{10.47} = 1.44$$

$$\theta = 55.3^\circ \text{ (using unrounded values)}$$

The magnitude of the resultant can be calculated using Pythagoras' theorem:

$$F_r = \sqrt{15.12^2 + 10.47^2} = 18.39 \text{ N}$$

So the resultant of this system of forces has magnitude 18.39 N and acts in a direction 55.3° below the horizontal with a positive sense.



► **Figure 1.26** Vector diagram showing how the sums of the vertical and horizontal components are combined to give the resultant

Conditions for static equilibrium

A system of forces in static equilibrium will have no tendency to move because all the forces acting in the system are perfectly balanced. The resultant force for a system in static equilibrium is zero. This also means that the horizontal and vertical components of all the forces are such that:

$$\Sigma F_v = 0$$

$$\Sigma F_h = 0$$

For a system of concurrent forces, these equations are enough to define static equilibrium. However, in a non-concurrent system of forces, where the lines of action do not intersect at the same point, there could be a tendency for the system to rotate.

The **moment** of a force describes the tendency of a force to produce rotation about a pivot point or centre of rotation.

Key terms

Equilibrant – the force that when applied to a system of forces will produce equilibrium. This force will be equal in direction and magnitude to the resultant but have the opposite sense.

Moment – the tendency of a force to rotate the object on which it acts.

The moment of a force (M) about a pivot is calculated by multiplying the magnitude of the force (F) by the perpendicular distance (s) from the pivot point to the line of action of the force.

$$M = Fs$$

So, for a system of non-concurrent forces, there are three conditions that must be met for static equilibrium:

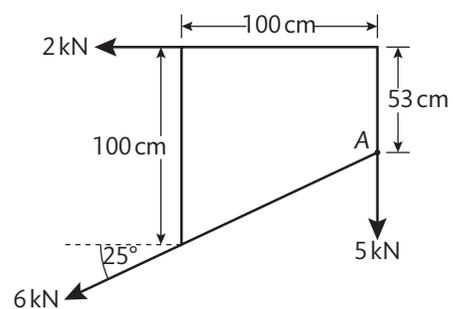
$$\Sigma F_v = 0$$

$$\Sigma F_h = 0$$

$$\Sigma M = 0$$

Worked Example

Find the equilibrant for the system of non-concurrent forces shown in **Figure 1.27**.



► **Figure 1.27** System of three non-concurrent coplanar forces

(Note that to define the equilibrant fully for a non-concurrent system of forces, you will need to find its magnitude, direction, sense and the perpendicular distance from its line of action to the centre of rotation of the body.)

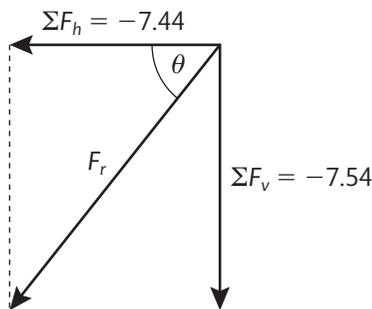
Solution

Find the sum of the vertical components:

$$\Sigma F_v = 0 - 6 \sin 25 - 5 = -7.54 \text{ kN}$$

Find the sum of the horizontal components:

$$\Sigma F_h = -2 - 6 \cos 25 + 0 = -7.44 \text{ kN}$$



► **Figure 1.28** Vector diagram showing how the sums of the vertical and horizontal components are combined to give the resultant

Figure 1.28 shows that the resultant F_r is acting in the negative sense in a direction θ below the horizontal, where:

$$\tan \theta = \frac{7.54}{7.44} = 1.01$$

$$\theta = 45.4^\circ$$

The magnitude of the resultant can be calculated using Pythagoras' theorem:

$$F_r = \sqrt{7.54^2 + 7.44^2} = 10.59 \text{ kN}$$

You can now say that the equilibrant has magnitude 10.59 kN, direction 45.4° above the horizontal and positive sense.

To complete the description of the equilibrant you must now take moments about some point in the system. This could be anywhere, but selecting point A in **Figure 1.27** will simplify the calculations because the moments of the 6 kN and 5 kN forces are zero about this point.

Before starting the calculations, you should establish the convention that clockwise moments are positive and anti-clockwise moments are negative.

Let the perpendicular distance from the equilibrant's line of action to point A be x m.

Taking moments about A:

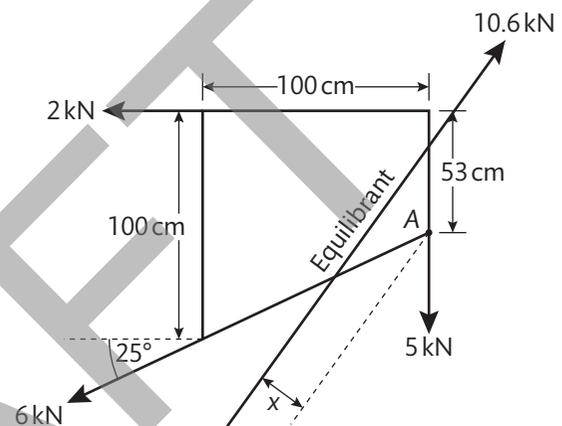
$$\Sigma M_A = (10.59 \times x) - (2 \times 0.53) + (6 \times 0) + (5 \times 0)$$

For equilibrium, $\Sigma M_A = 0$, so

$$10.59x = 2 \times 0.53$$

$$x = 0.1 \text{ m}$$

You can now fully define the equilibrant, which when applied to this system of forces would bring it into static equilibrium (see **Figure 1.29**).



► **Figure 1.29** The system of forces, showing the equilibrant that would bring it into static equilibrium

The equilibrant has magnitude 10.59 kN, direction 45.4° above the horizontal and positive sense, and the perpendicular distance from its line of action to point A is 0.1 m.

Simply supported beams

Beams have been used extensively for thousands of years to support structures and can be seen in everything from bridges and aircraft carriers to the playground see-saw.

When dealing with beams, we often need to consider two different ways in which they are loaded:

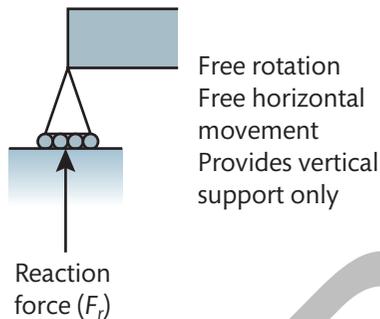
- **Concentrated loads** – a narrowly focused force that can be assumed to act at a specific point along the length of the beam; for example, the weight of a car parked on a bridge.
- **Uniformly distributed loads (UDL)** – a force distributed along the full length (or a defined section) of the beam, for example, the weight of the bridge itself.

Reactions

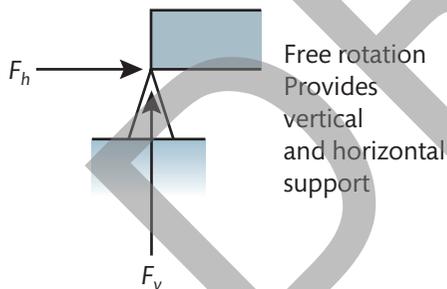
A simply supported beam is supported from below at two points, A and B. At each of these points, **support reaction** forces act on the beam to maintain the static equilibrium of the system. These can be calculated if we know the magnitude and position of the forces acting on the beam.

The type of support used dictates the direction in which the support reaction can act.

- ▶ **Rollers** provide a support **reaction normal** (perpendicular) to their point of contact with the beam. This is vertically upwards in the case of a horizontal beam (see **Figure 1.30**).
- ▶ **Pins** can provide a support reaction in any direction, so there can be a vertical and a horizontal component to the support reaction provided by a pinned joint (see **Figure 1.31**).



▶ **Figure 1.30** Roller support



▶ **Figure 1.31** Pin support

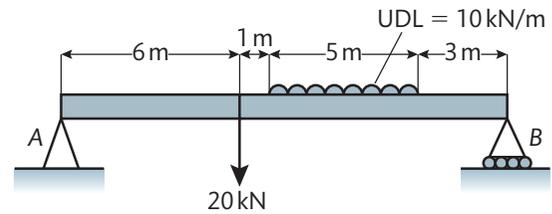
Key terms

Normal reaction – the force that acts perpendicular to a surface upon an object that is in contact with the surface.

Support reactions – the forces that are maintaining the equilibrium of a beam or structure.

Worked Example

Determine the support reaction for the simply supported beam shown in **Figure 1.32**.

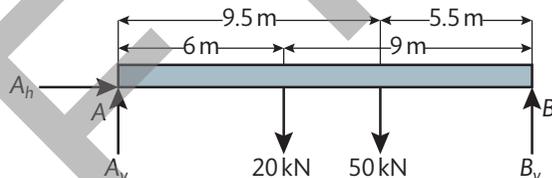


▶ **Figure 1.32** Space diagram of a simply supported beam

Solution

Draw a free body diagram of the beam, as in **Figure 1.33**.

(Note that the UDL can be treated as a point load acting at the centre of the distribution.)



▶ **Figure 1.33** Free body diagram of the simply supported beam

To find the support reaction B_v take moments about point A. Because the beam is in static equilibrium, you know that these moments must sum to zero.

$$\begin{aligned}\Sigma M_A &= (6 \times 20) + (9.5 \times 50) - (15 \times B_v) = 0 \\ B_v &= \frac{120 + 475}{15} = 39.67 \text{ kN}\end{aligned}$$

To find the support reaction A_v take moments about point B.

$$\begin{aligned}\Sigma M_B &= -(5.5 \times 50) - (9 \times 20) + (15 \times A_v) = 0 \\ A_v &= \frac{275 + 180}{15} = 30.33 \text{ kN}\end{aligned}$$

Now check that the other requirements for static equilibrium are met by considering the horizontal and vertical components of the forces acting on the beam. For equilibrium these must sum to zero.

Find the sum of the vertical components:

$$\Sigma F_v = A_v + B_v - 20 - 50 = 0$$

Find the sum of the horizontal components:

$$\Sigma F_h = A_h = 0$$

Once calculated and checked, state the solution clearly:

The support reaction at point A is 30.33 kN vertically upwards.

The support reaction at point B is 39.67 kN vertically upwards.



PAUSE POINT

Describe what is meant by static equilibrium in a non-concurrent system of forces.

Hint

Consider all the conditions that must be fulfilled for a body to be in static equilibrium.

Extend

Where a non-concurrent system is not in equilibrium, explain all the characteristics required to fully define the resultant and describe its relationship to the equilibrant.

B2 Loaded components

Table 1.9 summarises the formulae needed to calculate direct and shear loading.

► **Table 1.9** Summary of direct and shear loading

Operation	Direct loading	Shear loading
Stress	$\sigma = \frac{F}{A_\sigma}$	$\tau = \frac{F}{A_\tau}$
Strain	$\varepsilon = \frac{\Delta L}{L}$	$\gamma = \frac{a}{b}$
Elastic constants	Modulus of elasticity, $E = \frac{\sigma}{\varepsilon}$	Modulus of rigidity, $G = \frac{\tau}{\gamma}$

Forces acting on a body may be separated into two different types:

- **direct forces** – include tensile forces which tend to stretch and pull apart a material, and compressive forces (acting in the opposite direction) which tend to squash or compress a material.
- **shear forces** – forces cutting across a material which tend to shear or cut it apart.

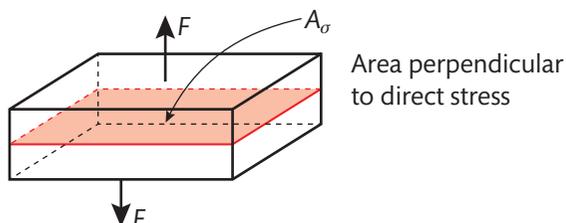
Stress

When either direct or shear forces act on a body, opposing reaction forces are distributed within the material and it is said to be stressed or under stress.

Stress is therefore a measure of the load distribution within a material and is expressed as the load carried per unit area of material. The unit of stress is newton per square metre (N m^{-2}) or pascal (Pa), where $1 \text{ N m}^{-2} = 1 \text{ Pa}$.

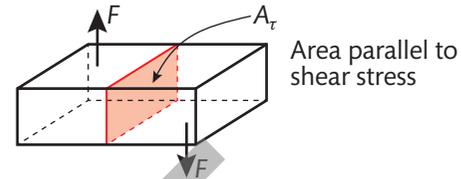
We can consider two types of stress – direct and shear.

- direct stress (σ) = $\frac{\text{normal force } (F)}{\text{area } (A_\sigma)}$ (see **Figure 1.34**)



► **Figure 1.34** The area A_σ considered in direct stress

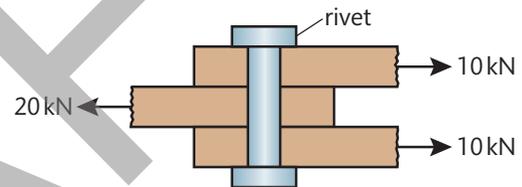
- shear stress (τ) = $\frac{\text{shear force } (F)}{\text{shear area } (A_\tau)}$ (see **Figure 1.35**)



► **Figure 1.35** The area A_τ considered in shear stress

Worked Example

A rivet has diameter (d) 20 mm and secures two rods together in the arrangement shown in **Figure 1.36**. Find the shear stress in the rivet.



► **Figure 1.36** Rivet in double shear

The rivet in this example is said to be in double shear. To solve this type of problem, consider either half the load acting on a single shear area (in this case the cross-sectional area of the rivet) or the whole load acting over the two shear areas (in this case twice the cross-sectional area of the rivet). First, calculate the cross-sectional area of the rivet in m^2 .

$$A = \pi \times \frac{d^2}{4} = \pi \times \frac{0.02^2}{4} = 3.14 \times 10^{-4} \text{ m}^2$$

In the first method, the shear area (A_τ) is the cross-section of the rivet and the shear force (F) is only half the applied load.

$$\begin{aligned} \tau &= \frac{F}{A_\tau} = \frac{10 \times 10^3}{3.14 \times 10^{-4}} \\ &= 31.8 \times 10^6 \text{ Pa or } 31.8 \text{ MPa} \end{aligned}$$

In the second method, the shear area (A_τ) is equal to twice the cross-section of the rivet and the shear force (F) is the whole of the applied load.

$$\begin{aligned} \tau &= \frac{F}{A_\tau} = \frac{20 \times 10^3}{2 \times 3.14 \times 10^{-4}} \\ &= 31.8 \times 10^6 \text{ Pa or } 31.8 \text{ MPa} \end{aligned}$$

Both methods give the same solution, which is that the shear stress in the rivet is 31.8 MPa.

Tensile and shear strength

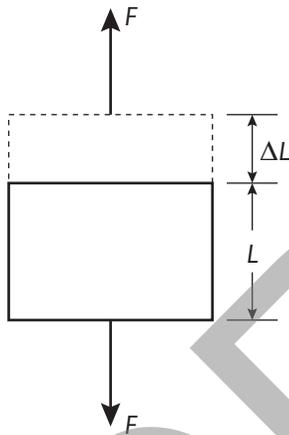
Tensile and shear strength are material-specific properties that specify the maximum tensile and shear stresses that can be applied to the material. If either the tensile or the shear strength of a material is exceeded, the material will rupture.

Strain

When a material is subjected to an externally applied stress, it has a tendency to change shape. Strain quantifies the deformation of a body as a proportion of its original length. As strain is the ratio of two lengths, it has no units.

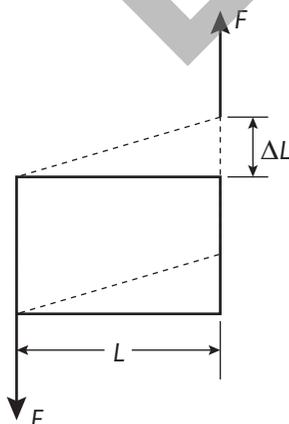
Again, we can consider two types of strain – direct and shear.

- ▶ direct strain (ϵ) = $\frac{\text{change in length } (\Delta L)}{\text{original length } (L)}$
(see **Figure 1.37**).



▶ **Figure 1.37** Direct strain

- ▶ shear strain (γ) = $\frac{\text{change in length } (\Delta L)}{\text{original length } (L)}$
(see **Figure 1.38**).



▶ **Figure 1.38** Shear strain

Worked Example

A press tool is required to punch out 40 mm diameter discs from a brass plate that is 1.5 mm thick and has a shear strength of 200 MPa.

- Calculate the force required to punch out the disc.
- Calculate the compressive stress in the punch.

Solution

- a) You are given the following values, which have been converted into standard units where necessary:

- diameter $d = 40 \text{ mm} = 0.04 \text{ m}$
- thickness $t = 1.5 \text{ mm} = 0.0015 \text{ m}$
- maximum shear strength
 $\tau_{\text{max}} = 200 \text{ MPa} = 200 \times 10^6 \text{ Pa}$

For the punch to cut through the brass, the maximum shear strength of 200 MPa must be applied over the shear area.

In this case, the shear area (A_r) will be equal to the circumference of the brass disc multiplied by its thickness:

$$A_r = \pi d \times t = \pi \times 0.04 \times 0.0015 \\ = 1.88 \times 10^{-4} \text{ m}^2$$

You know that $\tau = \frac{F}{A_r}$,

$$\text{so } F = \tau_{\text{max}} A_r = (200 \times 10^6) \times (1.88 \times 10^{-4}) \\ = 37\,600 \text{ N} = 37.6 \text{ kN}$$

The force required to punch out the brass disc is 37.6 kN.

- b) Compressive stress is a direct stress.

The area (A_σ) in this case will be the cross-sectional area of the punch:

$$A_\sigma = \frac{\pi d^2}{4} = \frac{\pi \times 0.04^2}{4} = 1.26 \times 10^{-3} \text{ m}^2$$

Using the formula for direct stress:

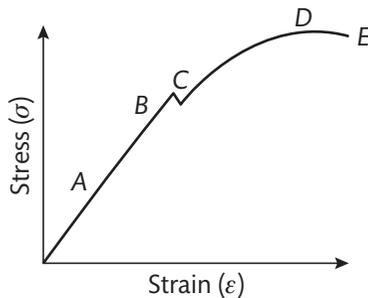
$$\sigma = \frac{F}{A_\sigma} = \frac{37\,600}{1.26 \times 10^{-3}} \\ = 29.8 \times 10^6 \text{ Pa} = 29.8 \text{ MPa}$$

The compressive stress in the punch is 29.8 MPa.

Elastic constants

A material is considered to perform in its **elastic region** while any change in strain brought about as a consequence of an applied stress will reduce back to zero once that stress is removed. **Figure 1.39** shows a typical stress–strain curve for a ferrous metal. The elastic region is that part of the curve below the yield point.

A Elastic region
B Limit of proportionality
C Yield point
D Ultimate tensile strength (UTS)
E Fracture point



► **Figure 1.39** Typical stress–strain curve for a ferrous metal

The **modulus of elasticity** (E), also known as **Young's modulus**, expresses the linear relationship between direct stress and direct strain exhibited by a material in the part of the stress–strain curve below the elastic limit. Its unit is newton per square metre (N m^{-2}).

$$\text{modulus of elasticity } (E) = \frac{\text{direct stress } (\sigma)}{\text{direct strain } (\epsilon)}$$

The **modulus of rigidity** (G) is a similar ratio, which expresses the linear relationship between shear stress and shear strain observed in many materials. Its unit is newton per square metre (N m^{-2}).

$$\text{modulus of rigidity } (G) = \frac{\text{shear stress } (\tau)}{\text{shear strain } (\gamma)}$$



PAUSE POINT

Explain the difference between shear stress and direct stress.

Hint

Use a sketch to illustrate each case.

Extend

Sketch a system in which a component is in both direct and shear stress.

Case study

The UK's longest suspension bridge

The Humber Bridge in Humberside is a suspension bridge crossing the Humber estuary. The bridge's structure relies on the strength of the two main cables used to support it.

Each of the main cables supporting the 1410 m central span is made up of bundles of 14 948 individual 5 mm diameter steel wires.

The steel used in the cables has an ultimate tensile strength (UTS) of 1540 MPa and a modulus of elasticity of 200 GPa.

When fully loaded, each of the main cables carries a load equivalent to 19 400 tonnes in tension.

The length of the unloaded cables was 1890 m (at 15°C) when they were manufactured.



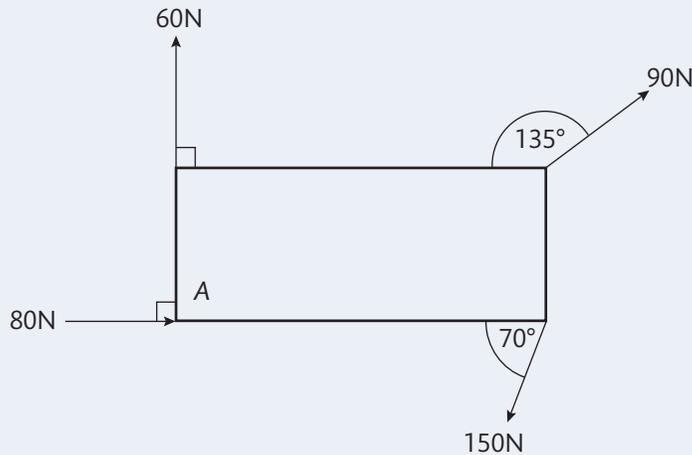
► **Humber Bridge**

Check your knowledge

- 1 Calculate the direct stress in each cable when it is fully loaded.
- 2 Calculate the corresponding direct strain for the cable.
- 3 Calculate the corresponding extension of the cable and its total length when in service.

Assessment practice 1.2

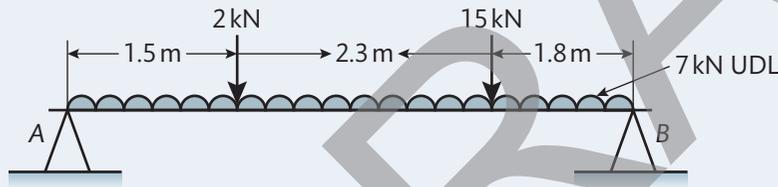
- 1 **Figure 1.40** shows a system of four non-concurrent coplanar forces acting on the corners of a rectangular plate measuring 3×8 m.



- **Figure 1.40** A system of four non-concurrent coplanar forces

Calculate the magnitude, direction and sense of the resultant force for this system, and find the position of its line of action relative to point A. (7 marks)

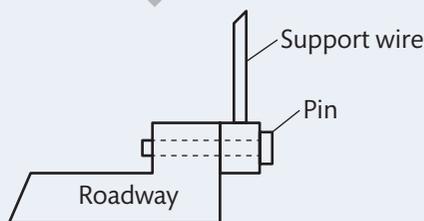
- 2 **Figure 1.41** shows a simply supported beam in static equilibrium.



- **Figure 1.41** A beam in static equilibrium

Calculate the support reaction forces at points A and B. (4 marks)

- 3 On a suspension bridge the vertical support wires attaching the roadway to the main suspension cables are each designed to support 2×10^6 N when the bridge is fully loaded. The support wires are connected to the roadway using pinned joints as shown in **Figure 1.42**. Each pin is 10^5 mm in diameter and has a shear modulus of 160×10^9 Pa.



- **Figure 1.42** Pinned joint connecting a support wire of the bridge to the roadway

- Calculate the shear stress in the pin when the bridge is fully loaded.
- Calculate the corresponding shear strain for the pin. (4 marks)

For each of the problems in this assessment practice, use the following stages to guide your progress through the task.

Plan

- What engineering principles will be involved in answering the question?
- Do I understand exactly what will be required to form a complete solution to the problem? Am I comfortable in how to approach each part?

Do

- I can organise my work logically and systematically, annotating my solutions with notes that explain my approach.
- I can use sketches effectively to help me visualise a problem and explain my approach to solving it.

Review

- I can give possible real-world applications of the principles used to solve the problem and explain why these are important in an engineering context.
- I can explain to others how best to approach similar problems.

C Dynamic engineering systems

C1 Dynamic engineering systems

As well as working with static systems, engineers are also interested in moving objects. This section introduces some of the basic principles and techniques that are used to understand dynamic systems.

Dynamic systems, by definition, are systems that involve the relative movement of several component parts or movement of the system as a whole. In this section, we will see how displacement, time, velocity and acceleration are related and how you can calculate the work done and the power required to overcome a resistance.

Kinetic parameters and principles

Kinetic parameters describe the uniform linear motion of an object over time. They include **displacement**, **velocity** and **acceleration**.

Key terms

Displacement – the straight-line distance between the start and finish positions of a moving object. Unit: metre (m).

Velocity – the rate at which the displacement of an object changes over time. Unit: metre per second (m s^{-1}).

Acceleration – the rate at which the velocity of an object changes over time. Unit: metre per second squared (m s^{-2}).

The relationships between the kinetic parameters that describe the motion of an object are defined using a set of equations that are based on the definitions of displacement, velocity and acceleration.

These are often referred to as the SUVAT equations from the letters used to represent the variables involved:

- ▶ Displacement (s) – the distance travelled by an object in time t .
- ▶ Initial velocity (u) – the starting velocity of an object when $t = 0$.
- ▶ Final velocity (v) – the final velocity of the object at time t .
- ▶ Acceleration (a) – the uniform acceleration of an object over time t .
- ▶ Time (t) – the period of time over which you will consider the motion of an object.

The SUVAT equations of motion are:

- ▶ $s = \frac{1}{2}(u + v)t$
- ▶ $s = ut + \frac{1}{2}at^2$
- ▶ $v = u + at$
- ▶ $v^2 = u^2 + 2as$.

Applying the SUVAT equations

A huge range of problems involving the linear motion of objects can be solved using the SUVAT equations.

Worked Example

A cyclist accelerates from 10 km h^{-1} to 35 km h^{-1} over 2000 m of track. Calculate the acceleration of the cyclist.

Solution

When approaching this problem, you should analyse the question carefully and identify the SUVAT variables that are mentioned:

$$s = 2000 \text{ m}, u = 10 \text{ km h}^{-1}, v = 35 \text{ km h}^{-1}, a = ?, t = \text{not given}$$

If you can identify three out of the five SUVAT variables you will be able to calculate the missing values.

Next, examine the units for each variable given in the question. Where these are not standard units, they **must** be converted before you begin your calculations. In this example both u and v are in km h^{-1} , so you need to convert them to m s^{-1} .

$$u = 10 \text{ km h}^{-1} = \frac{10 \times 1000 \text{ m}}{3600 \text{ s}} = 2.78 \text{ m s}^{-1}$$

$$v = 35 \text{ km h}^{-1} = \frac{35 \times 1000 \text{ m}}{3600 \text{ s}} = 9.72 \text{ m s}^{-1}$$

To choose the appropriate equation to solve this particular problem, find the one that includes only the variables involved. Here these are a , s , u and v , which corresponds to the SUVAT equation $v^2 = u^2 + 2as$.

Rearrange the equation to make a the subject:

$$a = \frac{v^2 - u^2}{2s}$$

To calculate a , substitute the numerical values from the question into the formula:

$$a = \frac{9.72^2 - 2.78^2}{2 \times 2000} = 0.022 \text{ m s}^{-2} \text{ (to 2 s.f.)}$$

The cyclist's acceleration is 0.022 m s^{-2} .

Dynamic parameters and principles

Dynamic parameters link the motion of an object to the forces involved in causing, influencing or stopping that motion. There are a number of important dynamic parameters. Their definitions and associated formulae are listed below.

Parameters of linear motion

Force (F) – a push or pull acting on an object. In physics a force is defined as any influence that tends to cause or alter the motion of an object. Unit: newton (N).

Static frictional force (F_s) – a force that must be overcome to set a sliding body in motion. The force (F_s) required to overcome static friction depends on the normal force (N) between the surfaces and their coefficient of static friction (μ_s). Unit: newton (N).

$$F_s = \mu_s N$$

Sliding (kinematic) frictional force (F_k) – a sliding body once in motion must overcome kinematic frictional resistance to its motion. The force (F_k) required to overcome kinematic friction depends on the normal force (N) between the surfaces and their coefficient of kinematic friction (μ_k). Unit: newton (N).

$$F_k = \mu_k N$$

Inertia force (F_i) – the resistance that an object of mass m has to any acceleration (a) that changes its state of motion. Unit: newton (N).

$$F_i = ma$$

Momentum (p) – the product of the mass (m) of a moving body and its velocity (v). Unit: kilogram metre per second (kg m s^{-1}).

$$p = mv$$

Work done (W) – the energy used when a force moves an object. It is the product of the applied force (F) and the associated displacement (s). Unit: newton metre (Nm) or joule (J).

$$W = Fs$$

Power (P) – the average rate of doing work (W) over time (t). Unit: joules per second (J s^{-1}) or watt (W).

$$P = \frac{W}{t} = \frac{Fs}{t}$$

Instantaneous power (P_i) – the product of force (F) and velocity (v). Unit: joules per second (J s^{-1}) or watt (W).

$$P_i = Fv$$

Weight (F_g) – the force exerted by a gravitational field (g) on a body with mass (m). Unit: newton (N).

$$F_g = mg$$

Gravitational potential energy (E_p) – the potential energy possessed by a body of mass m in a gravitational field (g) when raised to a vertical height (h). Unit: joule (J).

$$E_p = mgh$$

Kinetic energy (E_k) – the energy possessed by a body of mass (m) travelling with velocity (v). Unit: joule (J).

$$E_k = \frac{1}{2}mv^2$$

Newton's laws of motion

Newton's laws of motion are fundamental to our understanding of dynamics.

- ▶ Newton's first law of motion states that 'a body continues in its present state of rest or uniform motion in a straight line unless it is acted upon by an external force'.
- ▶ Newton's second law of motion states that 'the rate of change of momentum of a body is directly proportional to the resultant force that is producing the change'.
- ▶ Newton's third law states that 'to every acting force there is an equal and opposite reacting force'.

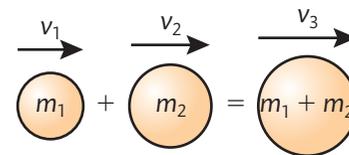
It is Newton's second law that underpins the derivation of an important equation that establishes the relationship between force (F), mass (m) and acceleration (a):

$$F = ma$$

Principle of conservation of momentum

The principle of conservation of momentum states that 'the total amount of momentum in a system remains constant unless the system is acted upon by an external force'.

Consider two bodies of known masses m_1 and m_2 moving in the same direction with velocities v_1 and v_2 , respectively, as shown in **Figure 1.43**. The two bodies then collide and begin to move together with velocity v_3 .



▶ **Figure 1.43** Conservation of momentum of two moving bodies that collide and then move together

Given that no external forces have been applied, the momentum prior to impact equals the momentum afterwards:

$$p_1 + p_2 = p_3$$
$$m_1v_1 + m_2v_2 = (m_1 + m_2)v_3$$



PAUSE POINT

Draw a diagram of a body in motion and annotate it to show how each of the parameters of linear motion is calculated.

Hint

As a starting point, you should give your body a mass, an acceleration, an instantaneous velocity and a displacement from a starting point.

Extend

Mark any external factors acting on the body that are required in order to determine the parameters of linear motion.

Case study

Flood defences

In 2016, many parts of the UK suffered serious flooding. The damage to property and infrastructure such as roads and bridges was severe, and the lives of many people were badly affected.

A common way of reinforcing river banks damaged by floodwater

is by using steel piles driven deep into the ground to prevent further erosion. The piles are installed side by side, forming a protective wall from the surface of the water down and through the river bed.



► **Hydraulic drop hammer** piling rig

Piles are installed using piling rigs (which are effectively large hammers) that are able to drive the piles vertically downwards. Each blow of the drop hammer moves the pile a little further into the ground.

The piling rig in the photograph has a 200 kg drop hammer, which is raised 3 m before being dropped onto the pile. It is inserting a pile that weighs 500 kg, which moves 125 mm further into the ground with each strike of the hammer.

Check your knowledge

- 1 What principles might you apply to the analysis of inserting a pile in this way?
- 2 Calculate the velocity with which the drop hammer hits the pile.
- 3 Calculate the initial velocity of the pile and hammer moving together.
- 4 Determine the force with which the ground is resisting the penetration of the pile.

Principle of conservation of energy

The principle of conservation of energy states that 'energy cannot be created or destroyed, but it can be changed from one form into another'.

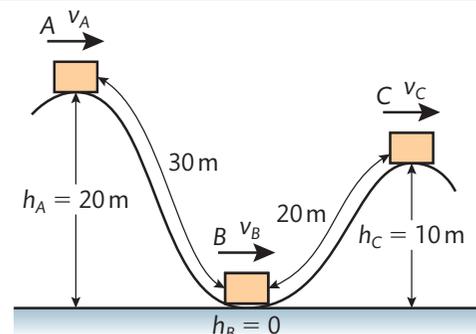
In dynamics this means that the total energy within a system always remains the same. Any decrease in one type of energy leads to a corresponding increase in another.

Worked Example

As shown in **Figure 1.44**, A rollercoaster car with mass 200 kg is stationary at point A before rolling down the track, passing point B at the bottom of the curve and then continuing up to point C at the top of the next curve. Throughout its motion along the track, the rollercoaster car has to overcome a frictional force of 200 N.

Using the principle of conservation of energy, calculate the velocities at points B and C.

► **Figure 1.44** The motion of a rollercoaster car along the rollercoaster track



Solution

By the principle of conservation of energy, the total energy in the system is the same at points A, B and C.

In this case the total energy (E_{total}) at each stage is made up of gravitational potential energy (E_p), kinetic energy (E_k) and work done to overcome friction (W):

$$E_{total} = E_p + E_k + W$$

$$E_{total} = mgh + \frac{1}{2}mv^2 + Fs$$

Consider point A where $h_A = 20$ m, $s_A = 0$ m, $v_A = 0$ m s⁻¹

Then

$$\begin{aligned} E_{total} &= (200 \times 9.81 \times 20) + \left(\frac{1}{2} \times 200 \times 0^2\right) + (200 \times 0) \\ &= 39\,240 + 0 + 0 = 39\,240\text{ J} \end{aligned}$$

At point A, all the parameters are defined, allowing us to establish that the total energy in the system is 39 240 J. At A this is made up entirely of gravitational potential energy.

Now that you know the total energy in the system, you can calculate the velocity of the rollercoaster car at point B, where

$$E_{total} = 39\,240\text{ J}, h_B = 0\text{ m}, s_{AB} = 30\text{ m}, v_B = ?$$

$$E_{total} = E_p + E_k + W$$

$$39\,240 = (200 \times 9.81 \times 0) + \left(\frac{1}{2} \times 200 \times v_B^2\right) + (200 \times 30)$$

$$39\,240 = 0 + 100v_B^2 + 6000$$

$$v_B = \sqrt{\frac{39\,240 - 6000}{100}} = \pm 18.23\text{ m s}^{-1}$$

The negative value would mean that the rollercoaster car is travelling backwards, so we use only the positive value, $v_B = 18.23$ m s⁻¹.

At point B, there is no longer any gravitational potential energy, 6000 J of work has been done to overcome the friction forces acting on the system, and the remaining energy has been transformed into kinetic energy now present in the moving rollercoaster car.

A similar procedure can be carried out to determine the velocity at point C, where

$$E_{total} = 39\,240\text{ J}, h_C = 10\text{ m}, s_{AC} = 50\text{ m}, v_C = ?$$

$$E_{total} = E_p + E_k + W$$

$$39\,240 = (200 \times 9.81 \times 10) + \left(\frac{1}{2} \times 200 \times v_C^2\right) + (200 \times 50)$$

$$39\,240 = 19\,620 + 100v_C^2 + 10\,000$$

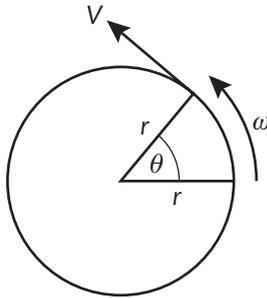
$$v_C = \sqrt{\frac{39\,240 - 19\,620 - 10\,000}{100}} = \pm 9.81\text{ m s}^{-1}$$

Again you want the positive value, so $v_C = 9.81$ m s⁻¹.

At point C, 19 620 J of gravitational potential energy has been retained in the system, a total of 10 000 J of work has been done to overcome friction forces, and the remaining energy has been transformed into kinetic energy present in the moving rollercoaster car.

Rotational motion

The fundamental parameters describing rotational motion are shown in **Figure 1.45**.



► **Figure 1.45** Basic parameters describing rotational motion

Angular displacement (θ) – the angle in radians through which a point or line has been rotated about a specific point (the centre of rotation). Unit: radian (rad).

Angular velocity (ω) – the rate at which angular displacement (θ) changes over time (t). Unit: radian per second (rad s^{-1}).

$$\omega = \frac{\Delta\theta}{\Delta t}$$

Angular acceleration (α) – the rate at which angular velocity (ω) changes over time (t). Units: radian per second squared (rad s^{-2}).

$$\alpha = \frac{\Delta\omega}{\Delta t}$$

Tangential velocity (v) – the linear velocity of a point moving in a circular path. All points in a rotating body share the same angular velocity (ω), but the tangential velocity of each point will depend on its distance from the centre of rotation (r). Unit: metre per second (m s^{-1}).

$$v = \omega r$$

Centripetal acceleration (a) – the linear acceleration acting on a rotating body towards its centre of rotation. It is defined in terms of the distance of the body from the centre of rotation (r) and either the angular velocity (ω) or the tangential velocity (v). Unit: metre per second squared (m s^{-2})

$$a = \omega^2 r = \frac{v^2}{r}$$

Torque (τ) – a turning force or moment that tends to cause rotational movement. It is defined by the sum force (F)

acting at a distance (r) from the centre of rotation. Unit: newton metre (Nm).

$$\tau = Fr$$

Work done (in uniform circular motion) (W) – the work done by a torque (τ) moving through an angular displacement (θ). Unit: joule (J).

$$W = \tau\theta$$

Power (in uniform circular motion) (P) – the rate of work done expressed as the product of torque (τ) and angular velocity (ω). Unit: watt (W).

$$P = \tau\omega$$

Moment of inertia (I) – sometimes called angular mass, is used in calculations relating to rotational motion in a similar way to which mass is used for linear motion. The moment of inertia for a particular rotating body is dependent on the mass distribution around the point of rotation and can be complicated to determine. Its calculation is beyond the scope of this unit.

Kinetic energy (of uniform circular motion) (E_k) – the energy possessed by a rotating body is calculated using its moment of inertia (I) and angular velocity (ω). Unit: joule (J).

$$E_k = \frac{1}{2}I\omega^2$$

Lifting machines

Lifting machines are used to allow relatively small forces to lift heavy objects. They include inclined planes, pulleys and scissor jacks. For instance, a person is unlikely to be able to lift and support a vehicle to change a wheel without using a lifting machine. Anyone can exert enough force to operate a scissor jack, which by virtue of a simple mechanism is able to amplify this force to a level sufficient to lift the car.

For all lifting machines, **mechanical advantage** (MA) is the ratio between the output force or load (F_l) and the input force or effort (F_e):

$$MA = \frac{F_l}{F_e}$$

II PAUSE POINT

Draw a diagram of a rotating body and annotate it to show how each of the parameters of rotational motion is calculated.

Hint

As a starting point, you should give your body a moment of inertia, an angular velocity and a distance from the centre of rotation.

Extend

Mark on the diagram any external factors acting on the body that are required in order to determine the parameters of rotational motion.

The **velocity ratio (VR)** is the ratio between the distance moved by the effort (s_e) and the distance moved by the load (s_l). Equivalently, it is the ratio of the velocity with which the effort moves (v_e) to the velocity with which the load moves (v_l):

$$VR = \frac{s_e}{s_l} = \frac{v_e}{v_l}$$

You already know that work done (W) is given by $W = Fs$. Applying this to the effort and load gives

$$W_e = F_e s_e$$

$$W_l = F_l s_l$$

The **efficiency** of a simple machine is given by the ratio of useful work output to work input.

Useful work output is the work done in actually moving the load. All machines will also have to overcome friction forces. Work done against friction is wasted as heat, noise or other undesirable effects.

The efficiency (η) of a simple machine with work input (W_e) and useful work output (W_l) is given by

$$\eta = \frac{W_l}{W_e} \times 100\%$$

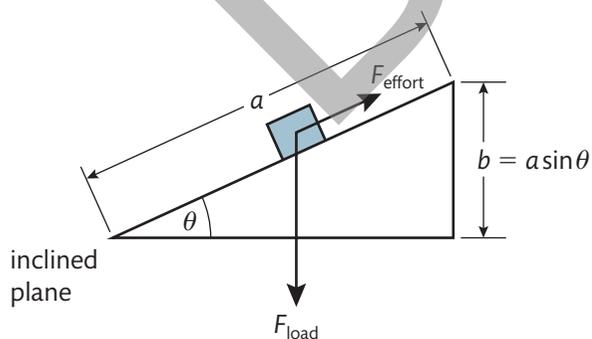
Efficiency can also be stated purely in terms of the mechanical advantage (MA) and the velocity ratio (VR):

$$\eta = \frac{MA}{VR} \times 100\%$$

Inclined plane

It is thought that in the absence of cranes and other modern machinery, the ancient Egyptians used an inclined plane (see **Figure 1.46**) to lift the enormous stones used to construct the pyramids. By pulling the load up an inclined plane sloping at an angle (θ), the effort moves a distance (a) while the load is lifted through a vertical distance (b).

$$VR = \frac{a}{b} = \frac{a}{a \sin \theta} = \frac{1}{\sin \theta}$$

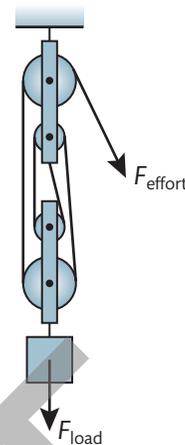


► **Figure 1.46** Inclined plane that can be used to lift a load

Pulleys

The velocity ratio of a pulley system is equal to the number of rope sections supporting the load.

Figure 1.47 shows a pulley system with $VR = 4$.

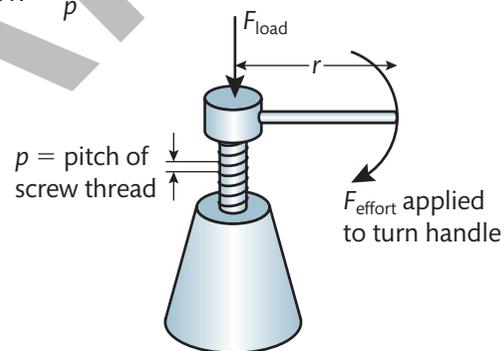


► **Figure 1.47** A system of pulleys with four rope sections supporting the load

Screw jack

In a screw jack like the one shown in **Figure 1.48**, one complete rotation of a handle of length (r) causes the load to move a distance equal to the pitch of the screw thread (p). The velocity ratio (VR) is given by

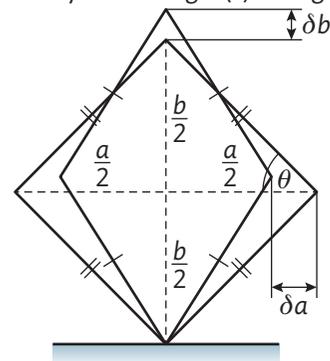
$$VR = \frac{2\pi r}{p}$$



► **Figure 1.48** Screw jack

Scissor mechanism

The velocity ratio of a scissor mechanism is not constant – it varies continuously as the angle (θ) changes.



► **Figure 1.49** Scissor mechanism

In the example shown in **Figure 1.49**, a small change in distance moved by the effort (δa) leads to a corresponding

(but not equal) small change in the distance moved by the load (δb). The velocity ratio depends on the geometry of the mechanism, but in this case

$$VR = \frac{\delta a}{\delta b} = \frac{1}{\tan \theta}$$

This means that:

- ▶ if $\theta < 45^\circ$ then $VR > 1$, so a movement of effort (a) will produce a larger movement in load (b)
- ▶ if $\theta = 45^\circ$ then $VR = 1$, so a movement of effort (a) will produce an equal movement in load (b)
- ▶ if $\theta > 45^\circ$ then $VR < 1$, so a movement of effort (a) will produce a smaller movement in load (b).

Scissor jack

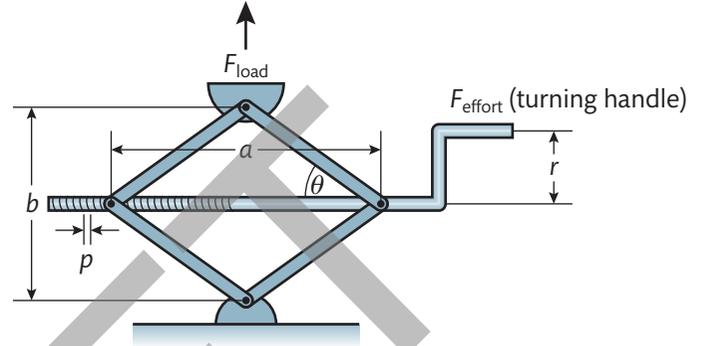
Car drivers often use a jack that incorporates a scissor mechanism when fitting the spare wheel to their vehicle.

Figure 1.50 shows a typical scissor jack, which uses a screw thread turned by a handle (just as in a screw jack, but here the screw does not lift the load directly) to

operate a scissor mechanism. For each rotation of the screw, the dimension a reduces by a distance equivalent to the pitch of the thread (p), and this motion is transferred by the scissor mechanism to increase the dimension b .

By combining the velocity ratios for the scissor and screw mechanisms, we get the velocity ratio for the scissor jack:

$$VR = \frac{1}{\tan \theta} \times \frac{2\pi r}{p} = \frac{2\pi r}{p \tan \theta}$$



▶ **Figure 1.50** Scissor jack that combines a screw jack with a scissor mechanism

Assessment practice 1.3

- 1 A car with mass 700 kg accelerates uniformly from rest to a velocity of 20 m s^{-1} in a time of 10 s. The frictional resistance to the motion of the vehicle is assumed to be constant throughout at 0.5 kN.
 - a) Calculate the force accelerating the car.
 - b) Calculate the average power developed by the engine. (2 marks)
- 2 A satellite launch vehicle travelling at 1000 m s^{-1} uses a controlled explosion to release its payload into orbit. The empty launch vehicle has a mass of 7500 kg and the satellite released has mass 250 kg. Both continue travelling in the same direction, with the launch vehicle now moving at 900 m s^{-1} .
 - a) Calculate the initial momentum of the launch vehicle immediately before the satellite was released.
 - b) Calculate the final velocity of the satellite after release. (2 marks)
- 3 A lift cage with mass 500 kg accelerates vertically upwards from rest to a velocity of 6 m s^{-1} in a distance of 12 m. The frictional resistance to motion is assumed to be constant throughout at 200 N.
 - a) Calculate the work done raising the lift.
 - b) Calculate the tension in the lifting cable.
 - c) Calculate the power developed by the winch. (3 marks)
- 4 A lorry with mass 3500 kg is parked at the top of a steep hill with a 1-in-8 gradient when its handbrake fails. Assume that the lorry has a constant frictional resistance to motion of 500 N. The lorry rolls 40 m down the hill before crashing into a lamp post.

Use the principle of conservation of energy to calculate the velocity of the lorry immediately prior to its impact with the lamp post. (3 marks)

For each of the problems in this assessment practice, use the following stages to guide your progress through the task.

Plan

- Which engineering principles need to be applied to this problem and what are the key formulae involved?
- How does the information given in the question dictate or otherwise influence my approach?

Do

- I have identified all the information I need to answer the question, including any formulae required.
- I have laid out my solution logically and explained each step so that my method can be followed easily.

Review

- I can identify the parts of my knowledge and understanding that require further development.
- I can identify the type and style of questions that I find most challenging and devise strategies, such as additional purposeful practice, that will help me to overcome any difficulties.

D

Fluid and thermodynamic engineering systems

D1 Fluid systems

There are many situations in which engineers must design ways to contain or transport fluids (liquids or gases), such as oil tankers, dams or pipelines. This topic will develop your knowledge of fluid dynamics as applied in a range of scenarios, including how water exerts a force on retaining walls, up-thrust and its effects on submerged objects, and flow in tapering pipes.



▶ Dam holding back a body of water

Fluid system parameters

The following are definitions of and formulae for the main parameters used to describe fluid systems.

Pressure (p) – a measure of force (F) distributed over an area (A). Unit: newton per square metre (N m^{-2}) or pascal (Pa), where $1 \text{ N m}^{-2} = 1 \text{ Pa}$.

$$p = \frac{F}{A}$$

Gauge pressure (p_{gauge}) – the pressure in a system relative to the ambient (or atmospheric) pressure. A pressure gauge is an instrument that measures the difference between the pressure of a contained fluid and its surroundings. If a pressure gauge on the surface of the Earth reads zero, it does not mean that there is a total vacuum or absence of pressure, only that the pressure being measured is the same as that of its surroundings (in this case atmospheric pressure). Unit: pascal (Pa).

Standard atmospheric pressure (p_{atm}) – the pressure on the surface of the Earth as a result of the weight of the atmosphere above our heads. Unit: pascal (Pa). Standard atmospheric pressure is defined as 101.325 kPa.

Absolute pressure (p_{abs}) – the pressure relative to a perfect vacuum. It is equal to gauge pressure plus standard atmospheric pressure. The equations describing

thermodynamic processes use absolute pressure, which takes atmospheric pressure into account. Unit: pascal (Pa).

$$p_{\text{abs}} = p_{\text{gauge}} + 101.325 \times 10^3 \text{ Pa}$$

Mass (m) – a measure of the amount of matter contained within a body. Unit: kilogram (kg).

Volume (V) – a measure of the amount of space occupied by a body. Unit: cubic metre (m^3).

Density (ρ) – describes the compactness of a material by measuring the amount of matter or mass (m) that is contained within a unit volume (V). Unit: kilogram per cubic metre (kg m^{-3}).

$$\rho = \frac{m}{V}$$

Weight (F_g) – the force exerted by a gravitational field (g) on a mass (m). Unit: newton (N).

$$F_g = mg$$

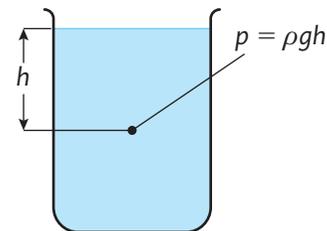
On Earth, the gravitational field strength (g) is usually taken as 9.81 N kg^{-1} .

Hydrostatic pressure

The hydrostatic pressure in an ocean increases with depth due to the weight of the water above. Submarines will be crushed if they descend too deep, and some deep-sea wrecks, such as the Titanic, can only be reached by specialist remotely operated vehicles.

The hydrostatic pressure (p) exerted by a column of fluid is dependent on the height (h) of the fluid column, the density of the fluid (ρ) and the gravitational field strength (g) (see **Figure 1.51**).

$$p = \rho gh$$

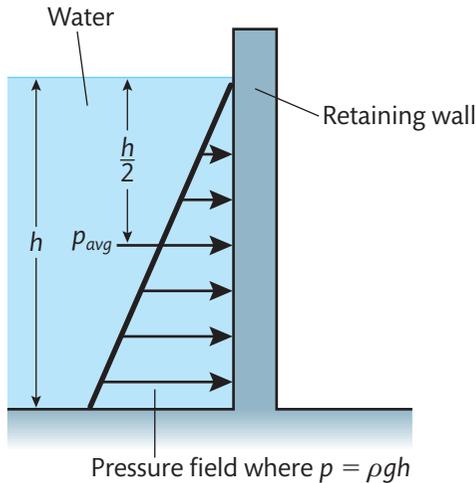


▶ **Figure 1.51** Hydrostatic pressure in a beaker of fluid

Notice that hydrostatic pressure is independent of the cross-sectional area of the fluid column. This means that the pressure felt at 10 cm depth in the Atlantic Ocean is the same as the pressure at 10 cm depth in your bath, which is also the same as the pressure at 10 cm depth in a tall glass (assuming they are all filled with sea water).

Submerged surfaces in fluid systems

Dams are used extensively in civil engineering projects to retain or redirect water in lakes or rivers (see **Figure 1.52**). It is vital that engineers build such structures strong enough to withstand the forces that are exerted by the weight of the water they retain.



► **Figure 1.52** A dam retaining a body of water

The pressure exerted on a submerged rectangular vertical surface, such as a dam, is not uniform over its entire submerged height (h). (Note that h is always measured downwards from the free surface of the fluid.)

Average pressure on a rectangular vertical surface

You know from $p = \rho gh$ that hydrostatic pressure is proportional to submerged height, increasing as h increases. Therefore:

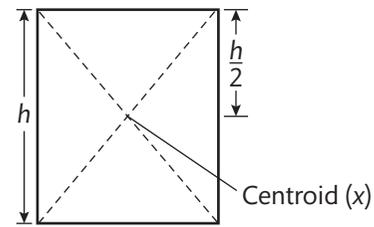
- minimum pressure will be at the free surface of the fluid, where $h = 0$ and so $p_{\min} = 0$
- maximum pressure will exist where $h = h$ and so $p_{\max} = \rho gh$.

This means that the average pressure (p_{avg}) exerted on the submerged surface will be:

$$p_{\text{avg}} = \frac{p_{\min} + p_{\max}}{2} = \frac{0 + \rho gh}{2} = \frac{\rho gh}{2}$$

An alternative way of determining p_{avg} is to consider the centroid of the submerged surface. This method is applicable to any shape of vertical submerged surface, not just rectangular.

The **centroid** or geometric centre of a shape is the average position of all the points in the shape. It is important because it determines the height at which the average pressure will act on a submerged plane surface. For a rectangular surface, finding the position of the centroid is straightforward (see **Figure 1.53**). For other geometric shapes it can be more complex.



► **Figure 1.53** The centroid of a rectangle is at the intersection of the diagonals

The average pressure on a submerged vertical surface will act at its centroid. For a simple rectangle the height of the centroid (x) is $\frac{h}{2}$, so:

$$p_{\text{avg}} = \rho gx = \frac{\rho gh}{2}$$

Hydrostatic thrust

If we know the average pressure (p_{avg}), then we can calculate the **average force** or **thrust** (F_T) acting on a submerged plane with surface area (A):

$$F_T = p_{\text{avg}} \times A$$

Therefore, the **hydrostatic thrust** (F_T) exerted by a fluid with density (ρ) acting on a submerged plane surface of area (A) whose centroid is at a distance (x) from the free surface of the fluid is given by:

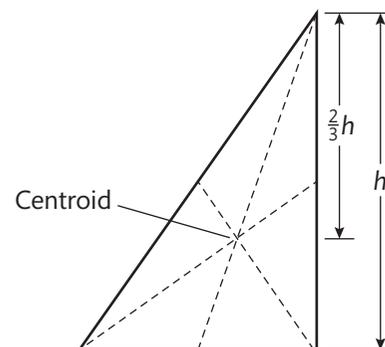
$$F_T = \rho gAx$$

Centre of pressure

Hydrostatic thrust can be treated as a single-point force acting at the **centre of pressure** of the submerged plane.

The centre of pressure for a rectangular submerged plane surface is determined by considering the centroid of the triangular pressure field shown in **Figure 1.54**. For a rectangular plane surface immersed vertically in a fluid, the distance (h_p) of the centre of pressure from the free surface is:

$$h_p = \frac{2}{3}h$$



► **Figure 1.54** Centroid of triangular pressure field



PAUSE POINT

Explain the difference between the height at which the average pressure on a submerged surface acts and the centre of pressure where the hydrostatic thrust can be considered to act.

Hint

Both values are determined by finding centroids.

Extend

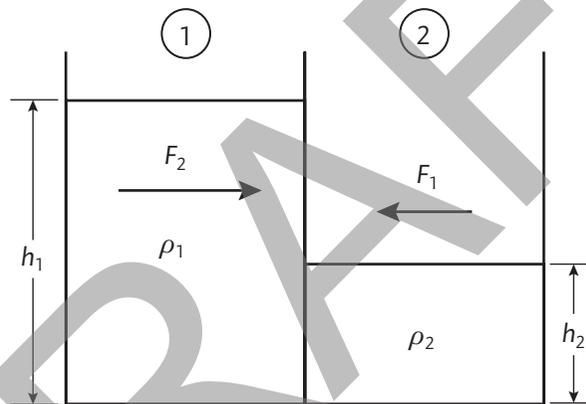
Taking the bottom of a dam as the centre of rotation, derive a formula to calculate the overturning moment acting on the dam as a consequence of the hydrostatic thrust exerted upon it.

Worked Example

A storage tank 2 m wide has a vertical partition across its width. Side 1 of the partition is filled with oil with a density of 900 kg m^{-3} to a depth of 1.8 m, and side 2 is filled with oil with a density of 750 kg m^{-3} to a depth of 0.9 m. Find the resultant hydrostatic thrust on the partition.

Solution

Draw a sketch like the one in **Figure 1.55**.



► **Figure 1.55** Sketch of the partitioned storage tank

Identify the parameters given in the question:

$$\rho_1 = 900 \text{ kg m}^{-3}, h_1 = 1.8 \text{ m}, A_1 = 1.8 \times 2 = 3.6 \text{ m}^2, \rho_2 = 750 \text{ kg m}^{-3}, h_2 = 0.9 \text{ m}, A_2 = 0.9 \times 2 = 1.8 \text{ m}^2$$

For a rectangular plane the average pressure will occur at a distance of $\frac{h}{2}$ from the free surface:

$$x = \frac{h}{2}$$

Hydrostatic thrust on the partition in side 1:

$$F_1 = \rho_1 g A_1 x_1 = 900 \times 9.81 \times 3.6 \times 0.9 = 28606 \text{ N}$$

Hydrostatic thrust on the partition in side 2:

$$F_2 = \rho_2 g A_2 x_2 = 750 \times 9.81 \times 1.8 \times 0.45 = 5960 \text{ N}$$

Let F_R be the resultant thrust acting from left to right. Then

$$F_R = F_1 - F_2 = 28606 - 5960 = 22646 \text{ N}$$

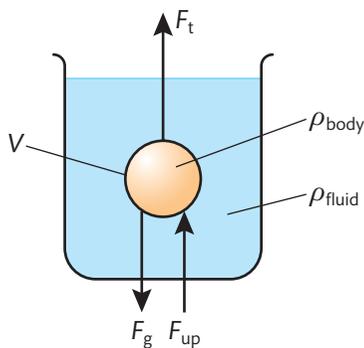
Immersed bodies

Archimedes' principle

Archimedes' principle states that 'a body totally or partially submerged in a fluid displaces a volume of fluid that weighs the same as the apparent loss in weight of the body'.

Submerged bodies

To help visualise Archimedes' principle, consider the suspended body submerged in a fluid shown in **Figure 1.56**. Given that the suspended body is in static equilibrium, there are three balanced forces acting upon it: the weight of the body (F_g) is balanced by the tension (F_t) in the wire supporting it and the up-thrust (F_{up}) according to Archimedes' principle.



► **Figure 1.56** The forces acting on a submerged body in static equilibrium

From the balance of forces we know that

$$F_t = F_g - F_{up}$$

where the weight (F_g) is given by

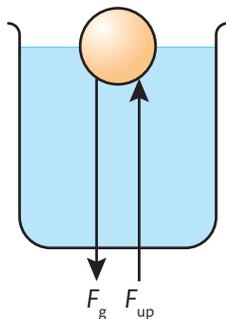
$$F_g = \rho_{body} V_{body} g$$

and Archimedes' principle tells us that

$$F_{up} = \rho_{fluid} V_{fluid} g$$

Determination of density using floatation methods

If a body is floating on the surface of a fluid (as in **Figure 1.57**), then there is no need to provide support to maintain static equilibrium, and so $F_t = 0$.



► **Figure 1.57** A partially submerged floating body in static equilibrium

In this case the balance of forces gives

$$F_g = F_{up}$$

and hence

$$\rho_{body} V_{body} = \rho_{fluid} V_{fluid}$$

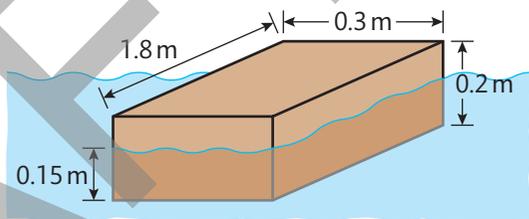
Remember that V_{fluid} is the volume of fluid displaced by the body and so is equal to only the volume of that part of the body which is submerged.

Worked Example

A block of wood 1.8 m long, 300 mm wide and 200 mm thick floats in seawater. The seawater has a density of 1020 kg m^{-3} and 150 mm of the block is below the free surface. Calculate the density and mass of the wood.

Solution

Draw a sketch like the one in **Figure 1.58**.



► **Figure 1.58** Block of wood partially submerged in seawater

Identify the parameters given in the question:

$$V_{body} = 1.8 \times 0.3 \times 0.2 = 0.108 \text{ m}^3$$

$$\rho_{fluid} = 1020 \text{ kg m}^{-3}$$

$$V_{fluid} = 1.8 \times 0.3 \times 0.15 = 0.081 \text{ m}^3$$

For a partially submerged body in static equilibrium:

$$\rho_{body} V_{body} = \rho_{fluid} V_{fluid}$$

$$\begin{aligned} \text{So } \rho_{body} &= \frac{\rho_{fluid} V_{fluid}}{V_{body}} = \frac{1020 \times 0.081}{0.108} \\ &= 765 \text{ kg m}^{-3} \end{aligned}$$

We also know that $\rho = \frac{m}{V}$, so

$$m_{body} = \rho_{body} V_{body} = 765 \times 0.108 = 82.6 \text{ kg}$$

Relative density

The **relative density** (d) of a substance is defined as the density of the substance compared with the density of pure water. It is given by

$$d_{\text{substance}} = \frac{\rho_{\text{substance}}}{\rho_{\text{water}}}$$



PAUSE POINT

Steel is much denser than water. Explain how it is possible that a steel ship can float.

Hint

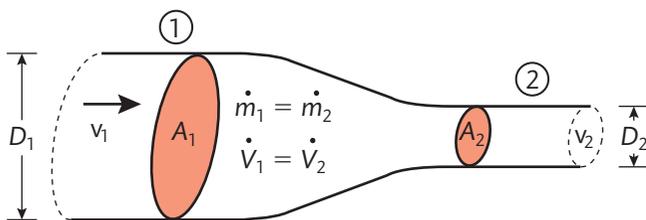
Consider the water displaced by a vessel.

Extend

A rowing boat carries a large rock to the centre of a lake. The rower then pushes the rock overboard. It sinks. At this point does the water level in the lake rise, fall or stay the same?

Fluid flow in a gradually tapering pipe

Figure 1.59 shows the flow of an incompressible fluid through a gradually tapering pipe. Analysis of the flow characteristics is simplified considerably if we assume that the density of the fluid, and so its volumetric and mass flow rates (defined below), all remain constant throughout.



► **Figure 1.59** Incompressible flow in a tapering pipe

Volumetric flow rate (\dot{V}) – the volume (V) of fluid that passes a given point in time (t). Unit: cubic metre per second ($\text{m}^3 \text{s}^{-1}$).

$$\dot{V} = \frac{V}{t}$$

For flow through a pipe, \dot{V} can be expressed in terms of the flow velocity (v) and the cross-sectional area of the pipe (A):

$$\dot{V} = Av$$

Mass flow rate (\dot{m}) – the mass (m) of fluid that passes a given point in time (t). Unit: kilogram per second (kg s^{-1}).

$$\dot{m} = \frac{m}{t}$$

Using the density of the fluid (ρ), the mass flow rate (\dot{m}) and volumetric flow rate (\dot{V}) are related by

$$\dot{m} = \rho \dot{V}$$

which can also be stated as

$$\dot{m} = \rho Av$$

Equations describing the continuity of flow

In a gradually tapering pipe as in **Figure 1.59**, the volumetric and mass flow rates are the same at points 1 and 2. This means that

$$\dot{V} = A_1 v_1 = A_2 v_2$$

and

$$\dot{m} = \rho A_1 v_1 = \rho A_2 v_2$$

Worked Example

On board a ship, seawater with a density of 1025 kg m^{-3} flows with a velocity of 6 m s^{-1} through a section of horizontal pipe with diameter 150 mm . The water is discharged over the side back into the sea at a velocity of 14 m s^{-1} through a gradually tapering nozzle fitted to the pipe.

- Calculate the volumetric flow rate in the pipe.
- Calculate the mass flow rate in the pipe.
- Calculate the final diameter to which the nozzle must be tapered.

Solution

Identify the parameters given in the question (referring to **Figure 1.59**):

$$\rho = 1025 \text{ kg m}^{-3}, v_1 = 6 \text{ m s}^{-1}, v_2 = 14 \text{ m s}^{-1}$$

$$A_1 = \frac{\pi D_1^2}{4} = \frac{\pi \times 0.15^2}{4} = 0.01767 \text{ m}^2$$

- $\dot{V} = A_1 v_1 = 0.01767 \times 6 = 0.106 \text{ m}^3 \text{ s}^{-1}$
- $\dot{m} = \rho \dot{V} = 1025 \times 0.106 = 108.68 \text{ kg s}^{-1}$ (using unrounded values in the calculation)
- $\dot{V} = A_1 v_1 = A_2 v_2$

$$\text{so } A_2 = \frac{\dot{V}}{v_2} = \frac{0.106}{14} = 0.00757 \text{ m}^2$$

$$A_2 = \frac{\pi D_2^2}{4}$$

$$\text{so } D_2 = \sqrt{\frac{4A_2}{\pi}} = \sqrt{\frac{4 \times 0.00757}{\pi}} = 0.098 \text{ m or } 98 \text{ mm}$$

D2 Thermodynamic systems

The transfer of heat can significantly affect the operational characteristics of some engineered components. In this section, you will see how the effects can be assessed and how to calculate the amount of heat energy required to complete certain processes.

Heat transfer parameters in thermodynamic systems

Temperature (T) – a measure of the kinetic energy of atomic or molecular vibrations within a body. Generally, temperature is measured in degrees Celsius ($^{\circ}\text{C}$), where 0°C is the freezing point and 100°C is the boiling point of water.

Thermodynamic temperature (T) is used in thermodynamic calculations and is measured on the absolute, or kelvin (K), temperature scale. Thermodynamic temperature has the same unit size as the Celsius scale, but the zero point for the kelvin scale is set at absolute zero. Absolute zero is a theoretical minimum possible temperature, at which the kinetic energy of molecular vibrations within a body would be zero. 0°C corresponds to 273 K, 20°C corresponds to 293 K and -20°C corresponds to 253 K.

Pressure (p) – a measure of force (F) distributed over an area (A). Unit: newton per square metre (N m^{-2}) or pascal (Pa), where $1 \text{ N m}^{-2} = 1 \text{ Pa}$.

Mass (m) – a measure of the amount of matter contained within a body. Unit: kilogram (kg).

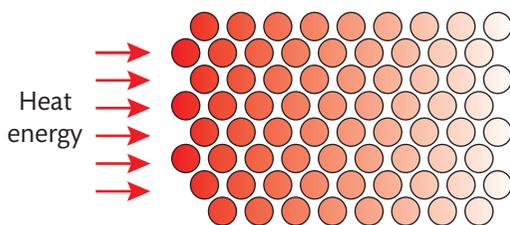
Thermal conductivity (λ) – a material property that describes a material's ability to conduct heat. Unit: watts per metre per kelvin ($\text{W m}^{-1} \text{K}^{-1}$).

Heat transfer processes

Heat transfer within or between bodies can occur by three distinct processes: conduction, convection and radiation.

Conduction

Conduction in solids and liquids involves the transmission of heat energy from one atom to another through physical contact (see **Figure 1.60**). As described previously, heat energy can be thought of as the kinetic energy of atomic vibrations. Atoms with high energy will pass some of this energy on to adjacent low-energy atoms, which establishes a flow of heat energy through the material. A secondary heat transfer process contributes to conduction, where free electrons moving through the material transfer heat energy. This goes some way towards explaining why good electrical conductors tend also to be good thermal conductors.



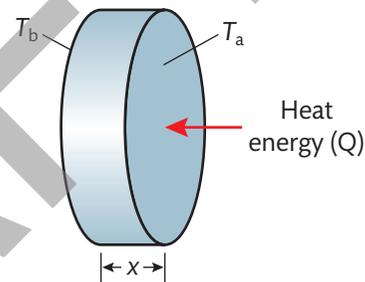
► **Figure 1.60** Heat transfer by conduction occurs when heat energy is passed from one atom to another through physical contact.

Conduction is more difficult in gases because the molecules or atoms are not in permanent contact. Heat energy can only be transferred during collisions between high-energy molecules or atoms and those with low heat energy. This helps to explain why gases are generally poor thermal conductors.

When conduction through a material (see **Figure 1.61**) is in steady state (with all variables remaining constant), the heat transfer rate (Q) can be expressed in terms of the surface area (A), temperature gradient ($T_a - T_b$), thickness (x) and thermal conductivity (λ) of the material:

$$Q = \frac{\lambda A(T_a - T_b)}{x}$$

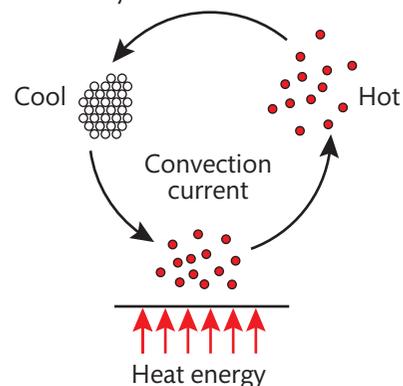
This formula shows that the rate of heat transfer by conduction (Q) is proportional to the surface area of the material (A). Often, the surface finish of a component designed for rapid heat transfer is textured so as to maximise its surface area.



► **Figure 1.61** Heat transfer by conduction through a material

Convection

Heat transfer by convection can occur only in liquids or gases where the molecules are free to move (see **Figure 1.62**). Any local heating in one part of such a material (caused by conduction or radiation) will cause localised expansion and a reduction in density. Fluid with lower density than its surroundings tends to rise, carrying its heat energy with it. Low-temperature fluid then flows in to replace the fluid that has risen and it, in turn, is warmed, expands and rises away. A convection current is established in this way.



► **Figure 1.62** Heat transfer by convection

Forced convection occurs when a fluid is forced to flow over a heat source in order to distribute heat energy to its surroundings. This is the principle employed in a hair dryer – cool ambient air is blown over an electric heating element and emerges at a much higher temperature.

Radiation

Radiation refers to the transfer of heat energy without physical contact. Instead, energy is transmitted in the form of electromagnetic waves (similar to those carrying light or radio signals), which are emitted as a result of energy changes in the orbits of electrons contained within the transmitting material.

This mechanism explains why on a hot sunny day we can feel the warmth of the sun, despite it being millions of miles away, through the vacuum of space.

Linear expansivity

A change in the temperature of a material is associated with a change in its size. This change in size acts in all directions. The amount by which the size changes given the same change in temperature differs from material to material and is defined by the material's **coefficient of linear expansion**.

Key term

Coefficient of linear expansion (α) – a material property that describes the amount by which the material expands upon heating with each degree rise in temperature. Unit: inverse kelvin (K^{-1}).

The change in length (ΔL) of a component with coefficient of linear expansion (α) and initial length (L) when subject to a temperature change (ΔT) is given by:

$$\Delta L = \alpha L \Delta T$$

II PAUSE POINT

Explain the three principal mechanisms of heat transfer.

Hint

Use diagrams to illustrate each process.

Extend

In terms of heat transfer processes, explain how a vacuum flask keeps hot things hot and cold things cold.

Worked Example

The wheels on a vintage railway carriage have steel tyres that have to be expansion-fitted to cast iron rims. The coefficient of linear thermal expansion for the steel used for the tyres is $13 \times 10^{-6} K^{-1}$. If the tyre has an internal diameter of 598.2 mm at $20^\circ C$ and the diameter of the wheel rim is 600 mm, determine the minimum temperature to which the tyre must be heated for it to be fitted.

Solution

Draw a sketch of one wheel as in **Figure 1.63**.

Identify the parameters given in the question:

$$\alpha = 13 \times 10^{-6} K^{-1}$$

$$L = 598.2 \text{ mm} = 0.5982 \text{ m at } 20^\circ C$$

$$\Delta L = 0.6 - 0.5982 = 0.0018 \text{ m}$$

$$T_1 = 20^\circ C$$

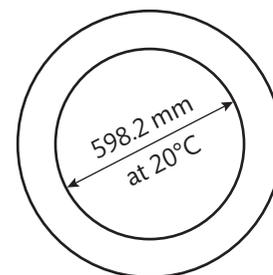
The important thing to realise here is that the size of the hole will expand in exactly the same way as if it were a solid disk of material.

You know that $\Delta L = \alpha L \Delta T$

$$\text{so } \Delta T = \frac{\Delta L}{\alpha L} = \frac{0.0018}{13 \times 10^{-6} \times 0.5982} = 231.46^\circ C$$

You know that $\Delta T = T_2 - T_1$

$$\text{so } T_2 = \Delta T + T_1 = 231.46 + 20 = 251.46^\circ C$$



► **Figure 1.63** A tyre of the railway carriage



PAUSE POINT

Why are gaps left between road sections in bridge constructions?

Hint

Consider what will happen to the bridge structure in warm weather.

Extend

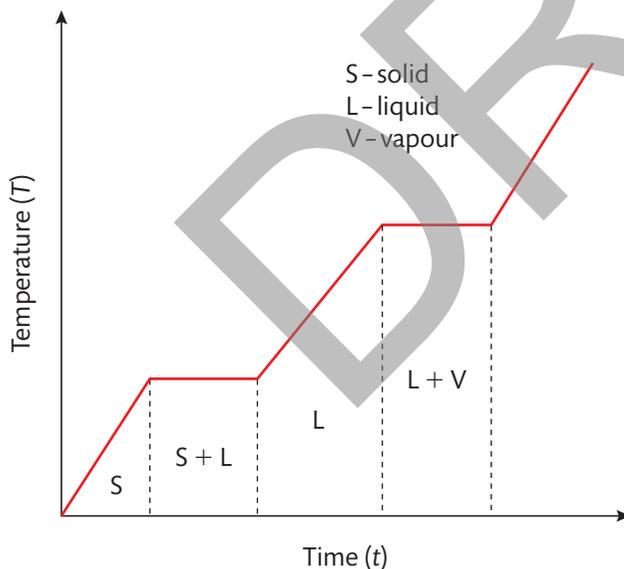
Research the operation of a bimetallic strip and explain its use.

Heat transfer and phase changes

If a solid material is subjected to a continuous input of heat energy, then its temperature will begin to rise until it reaches the melting point of the material (see **Figure 1.64**). The melting point of a solid is the temperature at which a **phase** change from solid to liquid will begin. During the phase change, continued heating will lead to no further increase in temperature until all of the solid has undergone the change into a liquid.

Once the phase change to a liquid is complete, continued heating will once again produce a rise in temperature, but at a different rate, until the boiling point of the liquid is reached. The boiling point of a liquid is the temperature at which a phase change from liquid to gas will begin. Again, during the phase change, continued heating will lead to no further increase in temperature until all the liquid has changed into a gas.

Once the phase change to a gas is complete, continued heating will once again produce a rise in temperature at yet another rate.



► **Figure 1.64** Change in temperature observed when heat energy is supplied to a material at a constant rate

Sensible heat

Heat energy that causes a change in the temperature of a material is called **sensible heat**. The amount of sensible

heat required to produce a given rise in temperature varies from material to material and even between the solid, liquid and gas phases of the same material.

Specific heat capacity (c) is a material property that describes the amount of heat energy required to raise the temperature of 1 kg of a material by one kelvin. Unit: joules per kilogram per kelvin ($\text{J kg}^{-1} \text{K}^{-1}$).

Sensible heat transfer (Q) is the product of the mass of material (m), the specific heat capacity of the material (c) and the change in temperature (ΔT). Unit: joule (J).

$$Q = mc\Delta T$$

Latent heat

Energy that does not cause a temperature rise but instead is absorbed by the material as it undergoes a phase change is called **latent heat**. For a given material, the latent heat of fusion (phase change from solid to liquid) will be different from the latent heat of vaporisation (phase change from liquid to gas).

Latent heat of fusion (L_f) is a material property that describes the amount of heat energy required for 1 kg of the material to undergo a change of phase from solid to liquid.

Latent heat of vaporisation (L_v) is a material property that describes the amount of heat energy required for 1 kg of the material to undergo a change of phase from liquid to gas.

Latent heat transfer (Q) is the product of the mass of material (m) and its latent heat (L). Unit: joule (J).

$$Q = mL$$

Key terms

Phase – the physically distinctive form of a substance: solid, liquid or vapour.

Sensible heat – heat energy that causes a change in the temperature of a substance.

Latent heat – heat energy causing a change of state of a substance without a change in temperature.

Worked Example

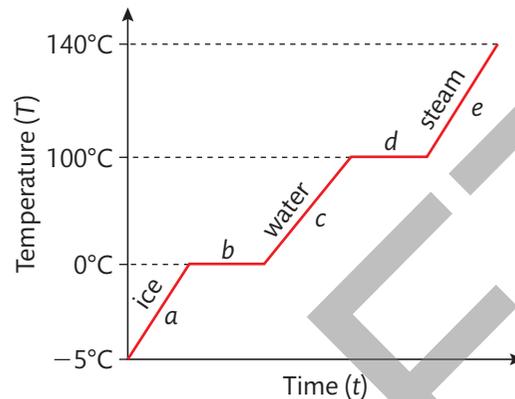
Calculate the energy required to convert 1 kg of ice at -5°C into superheated dry steam at 140°C .

For water, the latent heat of fusion is 334 kJ kg^{-1} , the latent heat of vaporisation is 2260 kJ kg^{-1} and the specific heat capacity is $4.2\text{ kJ kg}^{-1}\text{ K}^{-1}$.

The specific heat capacity of dry steam is $1.8\text{ kJ kg}^{-1}\text{ K}^{-1}$ and the specific heat capacity of ice is $2\text{ kJ kg}^{-1}\text{ K}^{-1}$.

Solution

Draw a sketch, like the graph in **Figure 1.65**, to show how the temperature of water changes with increasing heat energy.



► **Figure 1.65** Change in temperature when heat energy is supplied to water

Identify the parameters given in the question:

$$m = 1\text{ kg}, c_{\text{ice}} = 2\text{ kJ kg}^{-1}\text{ K}^{-1}, c_{\text{water}} = 4.2\text{ kJ kg}^{-1}\text{ K}^{-1}, c_{\text{steam}} = 1.8\text{ kJ kg}^{-1}\text{ K}^{-1}, L_f = 334\text{ kJ kg}^{-1}, L_v = 2260\text{ kJ kg}^{-1}$$

Considering each of the parts marked *a* to *e* on **Figure 1.65**:

$$Q_a = mc_{\text{ice}}\Delta T = 1 \times 2 \times 5 = 10\text{ kJ}$$

$$Q_b = mL_f = 1 \times 334 = 334\text{ kJ}$$

$$Q_c = mc_{\text{water}}\Delta T = 1 \times 4.2 \times 100 = 420\text{ kJ}$$

$$Q_d = mL_v = 1 \times 2260 = 2260\text{ kJ}$$

$$Q_e = mc_{\text{steam}}\Delta T = 1 \times 1.8 \times 40 = 72\text{ kJ}$$

So the total energy required is

$$Q_{\text{total}} = 10 + 334 + 420 + 2260 + 72 = 3096\text{ kJ}$$

Thermal efficiency of heat engines and heat pumps

Heat engines

Heat engines include internal combustion engines, steam engines and any other machine or process that changes heat energy into mechanical work.

In thermodynamic terms, a heat engine takes heat energy supplied at a high temperature and converts some of this energy into useful mechanical work while rejecting the remainder at some lower temperature:

$$\text{heat received at high temperature} = \text{work done} + \text{heat rejected to some lower temperature}$$

The **thermal efficiency** of a heat engine is the ratio of useful work done to the equivalent heat energy contained in the amount of fuel used:

$$\text{thermal efficiency} = \frac{\text{mechanical power output (W)}}{\text{equivalent heat energy of fuel (J kg}^{-1}\text{)} \times \text{fuel consumption rate (kg s}^{-1}\text{)}}$$

Worked Example

An engine under test developed an indicated power of 6.8 kW. Over the 30-minute test the engine used 780 g of fuel with an equivalent heat energy of 41.9 MJ kg⁻¹.

Calculate the thermal efficiency of the engine.

Solution

Identify the parameters given in the question:

$$\text{mechanical power} = 6.8 \text{ kW} = 6.8 \times 10^3 \text{ W}$$

$$\begin{aligned} \text{equivalent heat energy of fuel} &= 41.9 \text{ MJ kg}^{-1} \\ &= 41.9 \times 10^6 \text{ J kg}^{-1} \end{aligned}$$

$$\begin{aligned} \text{rate of fuel consumption} &= \frac{780 \times 10^{-3} \text{ kg}}{30 \times 60 \text{ s}} \\ &= 4.33 \times 10^{-4} \text{ kg s}^{-1} \end{aligned}$$

$$\begin{aligned} \text{So thermal efficiency} &= \frac{6.8 \times 10^3}{41.9 \times 10^6 \times 4.33 \times 10^{-4}} \\ &= 0.375 \end{aligned}$$

This means that only 37.5% of the potential heat energy in the fuel was converted into useful work.

Heat pumps

Heat pumps include systems such as refrigerators, air conditioning equipment and ground or air source heat pump central heating systems.

In thermodynamic terms, a heat pump performs in the opposite way to a heat engine because it takes in external energy to do work, extracts heat from a low temperature and delivers heat at a higher temperature:

$$\begin{aligned} &\text{external energy supplied} \\ &+ \text{heat extracted from a lower temperature} \\ &= \text{heat delivered at a higher temperature} \end{aligned}$$

In a heat pump used for refrigeration, the heat extracted from the low-temperature interior of a refrigerator is the useful output of the system. The performance of a refrigerator is not given in terms of an 'efficiency' but as a performance ratio (which can be greater than 1):

$$\begin{aligned} \text{refrigeration performance ratio} \\ &= \frac{\text{heat extracted}}{\text{external energy supplied}} \end{aligned}$$

In a heat pump that is used for heating, the heat delivered at a higher temperature is the useful output of the system. The performance of a heat pump is also expressed as a performance ratio:

$$\begin{aligned} \text{heat pump performance ratio} \\ &= \frac{\text{heat delivered}}{\text{external energy supplied}} \end{aligned}$$

Worked Example

A domestic ground source heat pump consumes 2 kW of electricity and can deliver 18 MJ of heat energy per hour to a house central heating system.

- Calculate the heat extracted from the ground per second.
- Determine the performance ratio of the heat pump when warming the house.
- Determine the performance ratio of the heat pump as a refrigerator cooling the ground.

Solution

Identify the parameters given in the question:

$$\begin{aligned} \text{external energy supplied} &= 2 \text{ kW} = 2000 \text{ W} \\ &= 2000 \text{ J s}^{-1} \end{aligned}$$

$$\text{heat delivered at higher temperature} = 18 \text{ MJ h}^{-1}$$

$$\begin{aligned} &= \frac{18 \times 10^6}{60 \times 60} \text{ J s}^{-1} \\ &= 5000 \text{ J s}^{-1} \end{aligned}$$

So

- heat extracted from a lower temperature (the ground) per second = 5000 – 2000 = 3000 J
- heat pump performance ratio = $\frac{5000}{2000} = 2.5$
- refrigeration performance ratio = $\frac{3000}{2000} = 1.5$.

Enthalpy and the first law of thermodynamics

Enthalpy (H) can be used to describe the total energy contained within a closed thermodynamic system. It includes any gravitational potential or kinetic energy.

The idea of enthalpy is reflected in the first law of thermodynamics, which states that 'the net energy supplied by heat to a body is equal to the increase in its internal energy and the energy output due to work done by the body'.

In other words, heat energy added into the system will either increase the internal energy (U) or be accounted for in the work done in expansion quantified by the product of pressure (p) and volume (V):

$$H = U + pV$$

Entropy and the second law of thermodynamics

Entropy (S) can be thought of as a measure of the dispersal of energy present in a thermodynamic system.

The second law of thermodynamics can be expressed in many ways, but it was first put forward in the form 'it is impossible for a self-acting machine unaided by an external agency to convey heat from one body to another at a higher temperature'. You have seen this when you looked at heat pumps. Heat flow would be spontaneous from a high temperature (with low entropy) to a low temperature (with higher entropy). However, in a heat pump, work must be done to reverse this natural process and move heat from a low temperature to a high temperature.

The second law can be expressed in terms of entropy as 'the entropy of any thermodynamic system and its surroundings will always tend to increase'.

This is the fundamental reason why heat flows from hot to cold temperatures and fluids flow from an area of high pressure to an area of low pressure.

The gas laws

Boyle's law states that provided the temperature (T) of a perfect gas remains constant, the volume (V) of a given mass of the gas is inversely proportional to the pressure (p) of the gas. This can be stated as

$$pV = \text{constant}$$

Charles's law states that provided the pressure (p) of a given mass of a perfect gas remains constant, the volume (V) of the gas will be directly proportional to the absolute temperature (T) of the gas. This can be stated as

$$\frac{V}{T} = \text{constant}$$

The **general gas equation** comes from combining Boyle's and Charles's laws and takes the form

$$\frac{pV}{T} = \text{constant}$$

Worked Example

Formula 1 tyres are designed to perform optimally at the full race temperature of 100°C when the gauge pressure in the tyre is 158 kPa . Tyre warmers are used before the race to pre-heat the tyres to 80°C . What pressure needs to be in the tyres when they are in the warmers to ensure optimal performance during the race?

Solution

During the race:

$$T_1 = 100^{\circ}\text{C} = 373\text{ K}$$

$$p_1 = 158\text{ kPa} = 158 \times 10^3\text{ Pa, which corresponds to an absolute pressure of } 259 \times 10^3\text{ Pa.}$$

In the tyre warmers:

$$T_2 = 80^{\circ}\text{C} = 353\text{ K}$$

$$p_2 = ?$$

These quantities are related by the general gas equation:

$$\frac{p_1 V_1}{T_1} = \frac{p_2 V_2}{T_2}$$

The tyre itself contains the gas in a constant volume, so $V_1 = V_2$. Eliminating these from the general gas equation gives

$$\frac{p_1}{T_1} = \frac{p_2}{T_2}$$

Make p_2 the subject of the equation: $p_2 = \frac{p_1 T_2}{T_1}$

Substitute in the known quantities:

$$p_2 = \frac{259 \times 10^3 \times 353}{373} = 245\,112\text{ Pa} \\ = 245\text{ kPa}$$

The pressure that should be set in the tyres when they are in the warmers is 245 kPa (absolute) or 144 kPa (gauge).

Assessment practice 1.4

- 1** A dam holding back water in a reservoir consists of a vertical retaining wall 10 m wide and 6 m high. The water has a density of 1000 kg m^{-3} and the water level is 1 m from the top of the wall. When designing and building dams it is essential that engineers understand the forces that will be acting on the dam so it can be made to withstand them.
- Calculate the resultant force exerted by the water on the dam.
 - Calculate the overturning moment acting around the base of the dam that the structure must be able to withstand.
- (8 marks)
- 2** 20 ml of air at atmospheric pressure of 101 kPa and temperature 20°C is contained in a fire piston with a piece of char cloth. The piston is struck and the air is rapidly compressed to 1 ml, causing a temperature increase to 420°C , which is sufficient to ignite the char cloth. Calculate the final pressure in the cylinder.
- (2 marks)
- 3** An aluminium bar with diameter 10 mm measures 1.2 m in length at 120°C . The bar is rigidly clamped at both ends before being allowed to cool to 20°C .
- The aluminium has a linear coefficient of expansion of $22.2 \times 10^{-6} \text{ }^\circ\text{C}^{-1}$, a tensile strength of 110 MPa and a Young's modulus of 69 GPa.
- Calculate the force induced in the aluminium bar during cooling.
 - Explain whether or not the bar will break during cooling.
- (5 marks)
- 4** A solid bronze cannon with mass 560 kg is being recovered from a shipwreck and is suspended by a lifting cable. The density of bronze is 8900 kg m^{-3} and the density of the seawater is 1025 kg m^{-3} .
- Calculate the tension in the lifting cable when it is suspending the fully submerged cannon.
 - Calculate the tension in the cable as the cannon leaves the water when it is held with a third of its volume still submerged.
- (5 marks)
- 5** In a water treatment plant, fresh water with density 1000 kg m^{-3} flows at 3 m s^{-1} through a horizontal section of pipe 60 mm in diameter. The section of pipe gradually tapers to a diameter of 90 mm.
- Calculate the volumetric flow rate in the pipe.
 - Calculate the mass flow rate in the pipe.
 - Calculate the flow velocity in the section of pipe where the diameter is 90 mm.
- (6 marks)

For each of the problems in this assessment practice, use the following stages to guide your progress through the task.

Plan

- How will I approach the task? Which principles and formulae are applicable?
- Do I need clarification around anything? Have I read and fully understood the question?

Do

- I understand my thought process and can explain why I have decided to approach the task in a particular way.
- I can identify where I have gone wrong and adjust my thinking to get myself back on course.

Review

- I can identify which elements I found most difficult and where I need to review my understanding of a topic.
- I can explain the importance of this area of learning in a wider engineering context.

E1 Static and direct current electricity

Static electricity

Static electricity is the build-up of electrical charge on an object. The word 'static' implies that there is no flow of electricity once the object is charged – that is, until the charge finds a suitable pathway through which to move or discharge. When you observe a lightning bolt, you have witnessed the discharge of static electricity from a storm cloud down to the ground or to another cloud. Static electricity is not always so spectacular, though – it is also generated when you rub a balloon on your hair or clothing, allowing you to stick the balloon to a ceiling.

Electrostatic forces are responsible for holding together the atoms and molecules that form the basis of everything that exists, including you.

Matter is made up of atoms. According to the Bohr atomic model, the core or nucleus of an atom contains neutrons and positively charged protons tightly bound together. The positive charge on the protons holds negatively charged electrons in orbit around the nucleus. Usually the positive charge possessed by the protons is balanced by the negative charge of the electrons held in orbit. However, electrons are only weakly held in orbit and have a tendency to move or migrate to adjacent atoms. An object becomes positively charged when electrons move away from it and too few remain to balance the positively charged protons. An object becomes negatively charged when it has an excess of electrons orbiting it, more than are required to balance the positively charged protons.

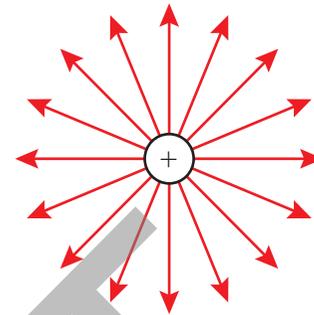
Electrical charge (q) is measured in coulombs (C). A single electron has an electrostatic charge of 1.602×10^{-19} C.

Electric fields

Wherever a charged particle or object is present, it generates an electric field acting in a particular direction. This can be represented using field lines (see **Figure 1.66**).

Wherever an electric field exists, any object with an electrical charge will experience a force acting upon it.

By convention, the arrows on the field lines indicate the direction of the force that would act on a **positive** test charge. The following are the important parameters used to describe electric fields.



► **Figure 1.66** Radial electric field lines from an isolated point charge

Electric field strength (E) – the strength at any point within an electric field, defined by the size of the force (F) that is exerted per unit of charge (q). Unit: newton per coulomb (N C^{-1}).

$$E = \frac{F}{q}$$

Electric flux (ψ) – measures the amount of flow of an electric field (as represented by electric field lines). The unit of electric flux is defined to be the amount of electric field emanating from a source with a positive charge of 1 C. Unit: coulomb (C), where a charge of 1 C gives rise to an electric flux of 1 C.

Electric flux density (D) – the amount of electric flux (ψ) passing through an area (A) that is perpendicular to the direction of the flux (as represented by the number of electric field lines per unit area). Unit: coulombs per square metre (C m^{-2}).

$$D = \frac{\psi}{A} = \frac{q}{A}$$

Coulomb's law

Where two objects are electrostatically charged, the electric fields they generate cause an electrostatic force to exist between them. The direction in which this force acts depends on the types of charges involved: like charges repel each other, while opposite charges attract. The magnitude of the force depends on the distance between the objects.

Coulomb's law is used to determine the magnitude of the electrostatic force that exists between two charged particles. It is expressed by the equation

$$F = \frac{q_1 q_2}{4\pi \epsilon_0 r^2}$$

where F is the force between two particles carrying electrostatic charges q_1 and q_2 that are held in free space with permittivity ϵ_0 at a distance r apart.

From the formula you can see that force and distance are related by an inverse square law, which means that the force will rapidly increase as the particles move closer together and rapidly decrease as they move apart.

Permittivity

Permittivity is a measure of the resistance of a material to the formation of an electric field. The force exerted between charged particles depends on the medium in which they are contained. The permittivity of free space (see below) is used in the Coulomb's law equation because this approximates to having the particles contained in air.

Absolute permittivity (ϵ) measures the ability of a material to resist the formation of an electric field within itself. It can also be thought of as a measure of how much electric flux is produced in a material by an electric field. Unit: farads per metre (F m^{-1}).

Permittivity of free space (ϵ_0) is the permittivity of a vacuum. For an electric field in a vacuum there is no medium that could affect the formation of electric flux. The permittivity of free space is a constant, $\epsilon_0 = 8.85 \times 10^{-12} \text{ F m}^{-1}$, and is the ratio of electric flux density (D) to electric field strength (E).

$$\epsilon_0 = \frac{D}{E}$$

Relative permittivity (ϵ_r) is used to define the permittivity of other media in comparison with that of a vacuum. ϵ_r is the ratio of a material's absolute permittivity (ϵ) to the permittivity of free space (ϵ_0).

$$\epsilon_r = \frac{\epsilon}{\epsilon_0}$$

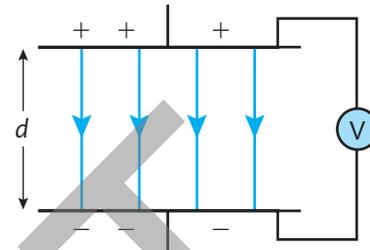
Uniform electric fields

Coulomb's law deals with the interaction of two point charges that have radial, and so diverging, field lines. In a uniform electric field, such as that produced between two parallel charged plates (see **Figure 1.67**), the field does not vary from place to place and the field lines are parallel and equally spaced.

Capacitance

In a uniform electric field between two electrically charged plates, as in **Figure 1.67**, the field strength (E) can be defined by the potential difference (V) or voltage between the plates and the distance (d) between them:

$$E = \frac{V}{d}$$



► **Figure 1.67** A uniform electric field between two plates forming a parallel plate capacitor

Capacitance (C) is a property of a pair of parallel charged plates. It defines the quantity of electrical charge (q) stored on the plates for each volt of potential difference (V) between them. Unit: farad (F).

$$C = \frac{q}{V}$$

The farad is defined as the capacitance when 1 V of potential difference corresponds to 1 C of electrical charge. It is a very large unit, so, in practice, capacitance is usually expressed in μF (10^{-6} F), nF (10^{-9} F) or pF (10^{-12} F).

Parallel plate capacitor

For a parallel plate **capacitor** like the one in **Figure 1.67**, the capacitance (C) is proportional to the area of the plates (A) and the permittivity (ϵ) of the material occupying the space between the plates (known as the **dielectric**) and is inversely proportional to the distance between the plates (d) (which also corresponds to the thickness of the dielectric):

$$C = \frac{\epsilon A}{d}$$

II PAUSE POINT

How does the permittivity of a dielectric material affect the capacitance of a parallel flat plate capacitor?

Hint

Consider absolute permittivity, relative permittivity and the permittivity of free space in your explanation.

Extend

Explain permittivity and its use in defining the relationship between electric field strength and electric flux density.

Current electricity

Current electricity deals with the flow of electrons around circuits in order to do work or process information.

Conductors and insulators

For an electric current to flow, a material must allow the free flow of loosely bound electrons between its atoms. Such materials are called electrical **conductors**. Materials called **insulators** have electrons that are more tightly bound to the atomic nucleus and so less free to move.

Electrical parameters

Electromotive force (e.m.f.) drives the movement of electrical charge in a closed circuit and causes current to flow from a point of high electrical potential energy to a point with lower electrical potential energy.

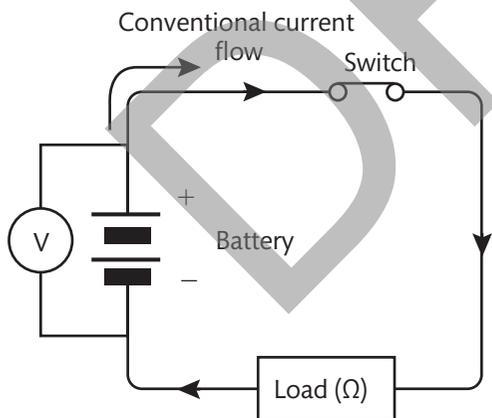
Potential difference (p.d.) is the difference in electrical energy that exists between two points in a circuit. It is usually referred to as voltage and is measured in volts (V).

Voltage (V) is the difference in electrical potential, or the potential difference, between two points in a circuit. Unit: volt (V).

Electrical current (I) is the rate of flow of electrical charge (q) in time (t). Unit: ampere (A), often abbreviated as 'amp'.

$$I = \frac{q}{t}$$

Current cannot flow in an open circuit because the electrons are unable to circulate. Once a circuit is completed, a conducting pathway has been provided and, by convention, current is said to flow from points with high electrical



► **Figure 1.68** Diagram illustrating measurement of circuit potential difference and conventional current flow

potential towards points of low potential (that is, from a high voltage to a low voltage). This is shown in the circuit diagram of **Figure 1.68**.

Resistance (R) describes the degree to which a circuit or component resists or opposes the free flow of an electric current. Unit: ohm (Ω), where 1Ω is defined as the resistance required so that a voltage of 1 V will result in a current flow of 1 A.

The reciprocal of resistance is **conductance** (G). Unit: siemen (S).

Resistivity (ρ) is a material property that describes the degree to which the material resists or opposes the free flow of electrons. Unit: ohm metre ($\Omega \text{ m}$).

Resistance and resistivity should not be confused. The resistance of a given sample of material is dependent on the size and shape of the sample. Resistivity is a property of the material itself and so is the same for samples of that material of any amount, size or shape. A helpful way to view the relationship between resistivity and resistance is to make a comparison with how density relates to mass.

► The resistance (R) of a wire with constant cross-sectional area (A) and length (l) made from a material with resistivity (ρ) is given by:

$$R = \frac{\rho l}{A}$$

The reciprocal of resistivity is **conductivity** (σ). Unit: siemens per metre (S m^{-1}).

Resistivity (and conductivity) of a material varies with temperature. In general, an increase in temperature will cause an increase in the resistivity of most conductors and a decrease in the resistivity of most insulators.

The **temperature coefficient of resistance** (α) for a material is defined as the increase in the resistance of a 1Ω resistor made from that material when its temperature is increased by 1°C .

► The change in resistance (ΔR) in a component with standard resistance (R_0) at 0°C and temperature coefficient of resistance (α) caused by a change in temperature (ΔT) is given by

$$\Delta R = \alpha \Delta T R_0$$

II PAUSE POINT

Explain the difference between resistance and resistivity.

Hint

Consider the resistance and resistivity of a length of copper wire and how these two characteristics are measured.

Extend

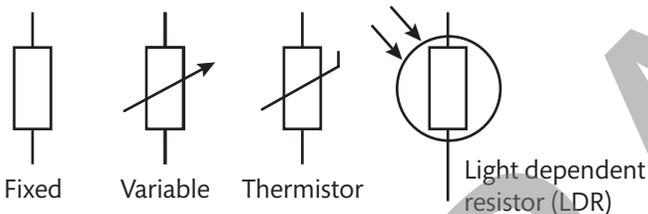
Explain the effects that temperature has on resistivity.

Resistors

Resistors are components used to limit the flow of current through a circuit. They are available in a wide range of values, from a fraction of an ohm to many mega ohms ($10^6\Omega$). There are several types of resistors, including wire wound, metal film and carbon resistors. The choice of which resistor to use in a circuit depends on usage and the power handling requirements – the latter might range from several watts for a wire wound resistor to a few milliwatts (mW) for a tiny surface-mount device used in a mobile phone.

- ▶ **Fixed resistors** are components designed to have a single specific value and are used extensively in electronic circuits.
- ▶ **Variable resistors** are resistors designed so that their resistance can be changed (such as a rotary volume control on an amplifier).

The circuit symbols for a fixed resistor and a variable resistor are shown in **Figure 1.69**.



▶ **Figure 1.69** Resistor circuit symbols

Capacitors

Capacitors are components designed to have a specific capacitance and so be able to store electrical charge.

Capacitors are used in a variety of forms in electronic circuits and are made from a number of different dielectric materials. **Table 1.10** lists different types of capacitors along with their capacitance range, working voltage and typical uses.

▶ **Table 1.10** Types of capacitors

Type	Capacitance range	Typical working voltage	Usage
Electrolytic	1 μF to 1 F	3V to 600V	Large capacitors; power supplies and smoothing
Tantalum	0.001 μF to 1000 μF	6V to 100V	Small capacitors; where space is restricted
Mica	1 pF to 0.1 μF	100V to 600V	Very stable; high-frequency applications
Ceramic	10 pF to 1 μF	50V to 1000V	Popular and inexpensive; general usage
Mylar	0.001 μF to 10 μF	50V to 600V	Good performance; general usage
Paper	500 pF to 50 μF	100V	Rarely used now
Polystyrene	10 pF to 10 μF	100V to 600V	High quality and accuracy; signal filters
Oil	0.1 μF to 20 μF	200V to 10 kV	Large, high-voltage filters

The circuit symbols for some capacitors are shown in **Figure 1.70**.



▶ **Figure 1.70** Capacitor circuit symbols

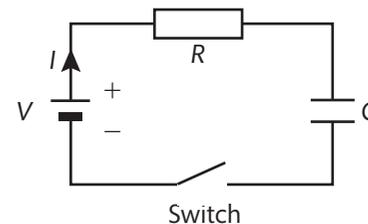
E2 Direct current circuit theory

The following are some key parameters used when working with capacitors.

- ▶ **Working voltage** – the voltage that can safely be applied to a capacitor without the dielectric breaking down (failing).
- ▶ **Charge stored** in a capacitor (q) – given by $q = CV$ for a capacitor with capacitance (C) connected to a voltage (V).
- ▶ **Energy stored** in a capacitor (W) – given by $W = \frac{1}{2}CV^2$ for a capacitor of capacitance (C) with an applied voltage (V).

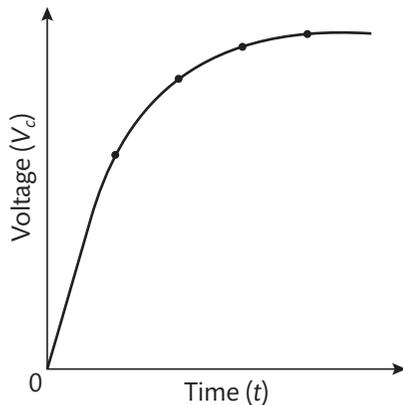
Charging a capacitor

Figure 1.71 shows a simple circuit where a direct current (d.c.) voltage is able to charge a capacitor (C) through a series resistor (R). When the switch is closed, current will flow until the capacitor is fully charged and reaches its steady state. Up to the point when current ceases to flow, the variations in circuit current and voltage across the capacitor and resistor are known as **RC transients**.



▶ **Figure 1.71** A capacitor-charging circuit, with a capacitor and a resistor connected in series

If the voltage across the capacitor (V_C) is plotted against time (t), as in **Figure 1.72**, the graph shows that the growth in voltage is exponential, characterised by the curve rising steeply before flattening out as it approaches its maximum value.



► **Figure 1.72** Voltage across a charging capacitor

At the moment when a capacitor starts to charge, the rate of growth of the voltage is high. If the voltage growth were maintained at this initial rate, then the time it would take for the voltage to reach its maximum value is known as the **time constant**.

In fact, more generally, the time constant for an exponential transient can be defined as the time taken for a transient to reach its final value from the time when its rate of change is maintained.

In a series-connected capacitor and resistor circuit, the value of the time constant (τ) is equal to the product of the capacitance (C) and the resistance (R):

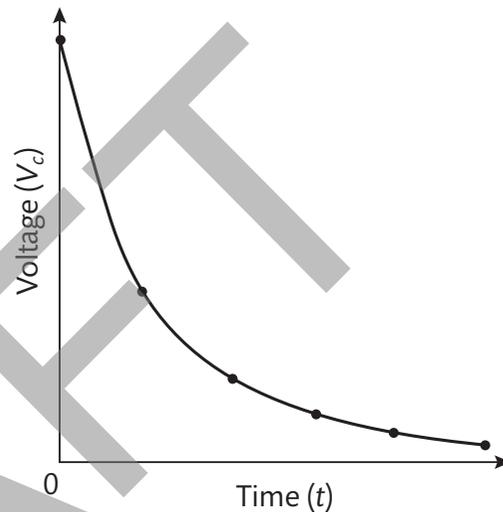
$$\tau = RC$$

When a capacitor with capacitance (C) is charged through a resistance (R) towards a final potential (V_0), the equation giving the voltage (V_C) across the capacitor at any time (t) is the **capacitor charge equation**:

$$V_C = V_0(1 - e^{-t/RC})$$

Discharging a capacitor

Once fully charged, if the power supply in the capacitor-charging circuit is replaced by a short circuit, then current will flow out of the capacitor as it discharges. If the voltage (V_C) across the capacitor is plotted against time (t), the graph will show that the reduction or decay in voltage is exponential. Exponential decay is characterised by a curve falling steeply before flattening out as it approaches its minimum value (often zero), as shown in **Figure 1.73**.



► **Figure 1.73** Voltage across a discharging capacitor

When a capacitor with capacitance (C) is discharged through a resistance (R) from an initial potential (V_0), the equation giving the voltage (V_C) across the capacitor at any time (t) is the **capacitor discharge equation**:

$$V_C = V_0 e^{-t/RC}$$

Link

See the section on 'Exponential growth and decay' in **A1 Algebraic methods** for worked examples of problems.

Note that when a capacitor is disconnected from a power source, it might retain its charge for a considerable length of time. It is good practice, therefore, to connect a high-value resistor in the circuit across the capacitor terminals. This ensures that the capacitor will be automatically discharged once the supply is switched off.

II PAUSE POINT

Explain the meaning of the time constant in the exponential equations for the charge and discharge of capacitors.

Hint

What are the two parameters used to calculate the time constant?

Extend

Draw a sketch of an exponential function and explain the relationship between the tangent to the curve at any point and the time constant.

Ohm's law

The relationship between voltage (V) and current (I) at constant temperature obeys **Ohm's law**:

$$\frac{V}{I} = \text{a constant}$$

This constant is called the resistance (R), and the relationship can be expressed as:

$$V = IR$$

Using Ohm's law, electrical power (P) can be expressed in terms of voltage (V), current (I) and resistance (R):

$$P = IV = I^2R = \frac{V^2}{R}$$

Its unit is the watt (W).

The **power efficiency** (E) of a system is the ratio between the power output (P_{out}) and the power input (P_{in}):

$$E = \frac{P_{\text{out}}}{P_{\text{in}}}$$

Ohm's law can be used to analyse a simple circuit, but in more complex situations involving networks of resistors, Kirchhoff's laws are more useful.

- ▶ **Kirchhoff's voltage law** – in any closed loop network, the total p.d. across the loop is equal to the sum of the p.d.s around the loop. For example, if three resistors are connected in series, with voltage drops (V_1 , V_2 and V_3) across them, then the total supply voltage (V) is given by

$$V = V_1 + V_2 + V_3$$

For n resistors connected in series, using $V = IR$ for each resistor we get:

$$V_1 + V_2 + V_3 + \dots + V_n = IR_1 + IR_2 + IR_3 + \dots + IR_n$$

which can also be expressed as:

$$\Sigma \text{p.d.} = \Sigma IR$$

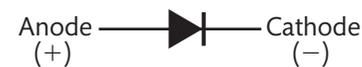
- ▶ **Kirchhoff's current law** – at any junction of an electric circuit, the total current flowing towards the junction is equal to the total current flowing away from the junction. For example, if three resistors are connected to a common point, with currents (I_1 , I_2 and I_3) flowing in towards the junction and current (I) flowing away from the junction, then:

$$I = I_1 + I_2 + I_3$$

Diodes

A diode is a very common electronic component made of a semiconductor material. It allows current to flow in only one direction. The circuit symbols for three types of diodes are shown in **Figure 1.74**. The direction of the arrow indicates the direction in which current can flow through the diode.

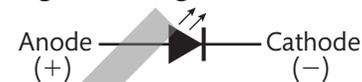
Semiconductor diode



Zener diode



Light-emitting diode



▶ **Figure 1.74** Diode symbols

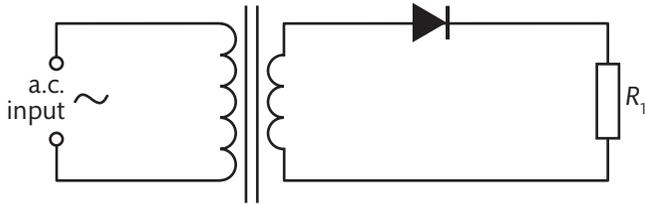
When a voltage is applied to a diode and a current flows in the direction of the arrow, the diode is said to be **forward biased**. With a material such as silicon, a forward bias of approximately 0.7V is required for a diode to begin to allow current to flow.

Generally, a **reverse bias** of any voltage will result in no current flowing through the diode, with the exception of the Zener diode (see **voltage regulation**).

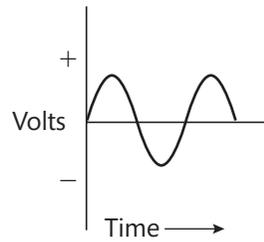
Forward bias applications

- ▶ **Rectification** – one important application of diodes is in the rectification of an alternating current (a.c.) electricity supply into a direct current (d.c.) output. Electronic devices tend to require relatively low d.c. voltages to operate. As the domestic electricity supply in the UK is 230V a.c., a step-down transformer is used first to reduce the voltage to the required level, usually between 3V and 15V depending on the application. The output of the transformer is still a.c., so a rectifier circuit must be used to convert it to d.c. **Figure 1.75** shows examples of simple rectifier circuits. A capacitor is then connected across the output to smooth the varying voltage to approximate a d.c. voltage.
- ▶ **Component protection** – diodes are also commonly used to protect electronic devices such as integrated circuits (ICs) or transistors from transient voltage spikes induced in relay coils when they are switched off. The current flowing through a relay coil generates a magnetic field that quickly collapses when the current is turned off. The collapsing field induces a back e.m.f. in the coil, which if allowed to build might cause damage to the electronic components. A diode connected across the coil conducts the current away from the circuit components (see **Figure 1.76**).

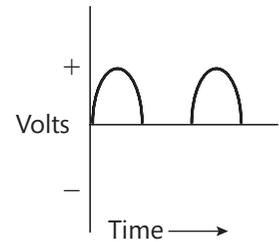
Half-wave (1 diode)



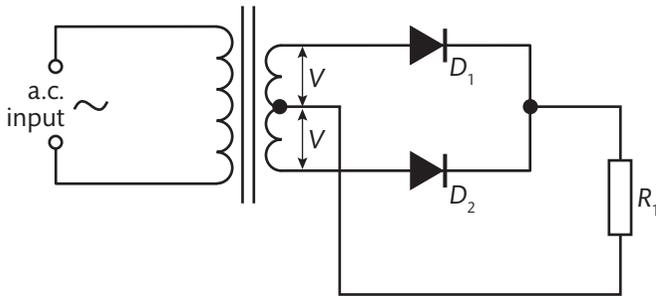
Input



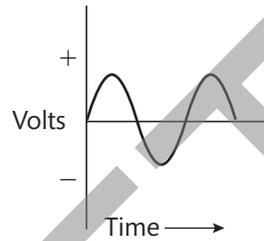
Output



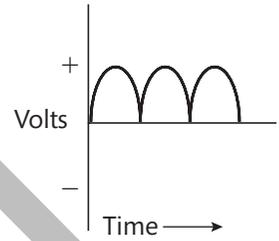
Bi-phase full-wave (2 diodes, tapped transformer)



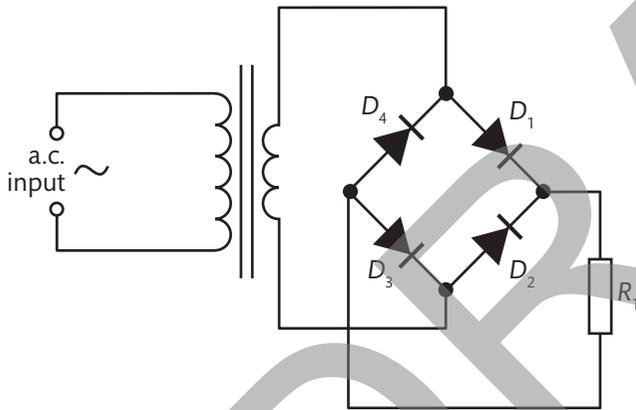
Input



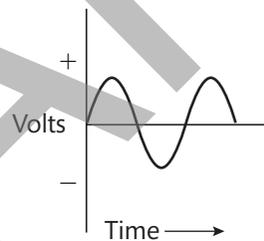
Output



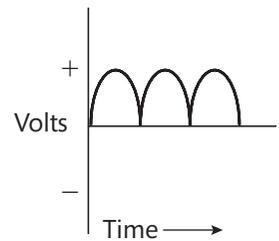
Full-wave (4 diodes)



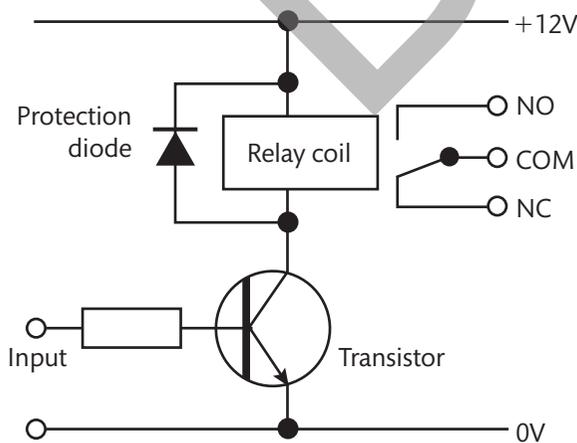
Input



Output



► **Figure 1.75** Simple rectifier circuits



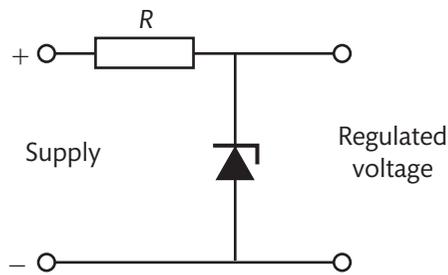
► **Figure 1.76** Use of a protection diode

Link

The generation of a back e.m.f. by a collapsing magnetic field is an example of self-inductance, which is dealt with in more detail in the section on 'Self-inductance in a coil' under **F Magnetism and electromagnetic induction**.

Reverse bias applications

- **Voltage regulation** – a Zener diode is constructed such that when a reverse bias voltage reaches a defined level, the diode begins to allow the flow of current. The point at which the diode starts to conduct in reverse bias is called the breakdown voltage. This can be set at anywhere from 2.7V to 150V. The characteristics of Zener diodes allow them to be used in voltage regulator circuits, which are able to maintain a constant voltage output. A simple voltage regulator circuit using a resistor and a Zener diode is shown in **Figure 1.77**.



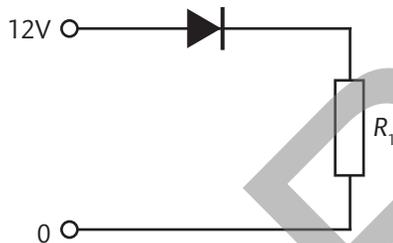
► **Figure 1.77** A simple voltage regulator circuit using a Zener diode

Series resistors and diodes

Where a forward bias diode is connected in series with a resistor, it is important to remember that there will be a voltage drop across the diode. For silicon diodes this is typically 0.7V.

Worked Example

Consider the circuit shown in **Figure 1.78**, where a silicon diode and a resistor are connected in series. Given that the supply voltage is 12V d.c. and $R_1 = 4.7 \text{ k}\Omega$, find the current flowing in the circuit.



► **Figure 1.78** Resistor connected in series with a diode

Solution

The voltage drop across a forward bias silicon diode is 0.7V. Applying Kirchhoff's voltage law, you can determine the voltage drop across resistor R_1 :

$$V_T = V_1 + V_2$$

$$V_1 = 12 - 0.7 = 11.3\text{V}$$

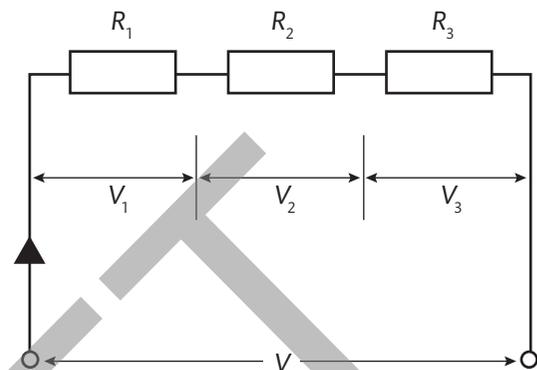
You can now determine the current in R_1 , which is the current flowing in the whole series circuit, by using Ohm's law:

$$I = \frac{V}{R} = \frac{11.3}{4.7 \times 10^3} = 0.0024\text{A or } 2.4\text{mA}$$

Resistors in series and parallel

Resistors in series

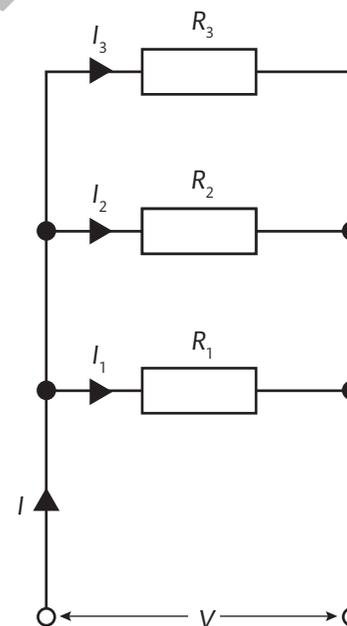
Figure 1.79 shows three resistors connected in series. In this configuration it is important to remember that the same current flows throughout the circuit, and so the current flowing through each resistor will be the same.



► **Figure 1.79** Resistors connected in series

Resistors in parallel

Figure 1.80 shows three resistors connected in parallel. In this configuration, it is important to note that the voltage across each resistor will be the same.



► **Figure 1.80** Resistors connected in parallel

By applying Ohm's and Kirchhoff's laws, you can show that the total resistance (R_T) for a number (n) of resistors in parallel is:

$$\frac{1}{R_T} = \frac{1}{R_1} + \frac{1}{R_2} + \dots + \frac{1}{R_n}$$

E3 Direct current networks

It is common for direct current (d.c.) circuits to have a combination of series and parallel elements.



PAUSE POINT

Recall the three laws used extensively to analyse d.c. circuits.

Hint

Give the name of each law and define it both in words and as a mathematical formula.

Extend

Using a combination of these laws, derive the formulae for determining the total resistance for a number of resistors in series.

Series and parallel combinations

If a circuit contains resistors that are connected in a combination of series and parallel configurations, then it is necessary to use a combination of the series and parallel formulae to calculate the various current, voltage and resistor values.

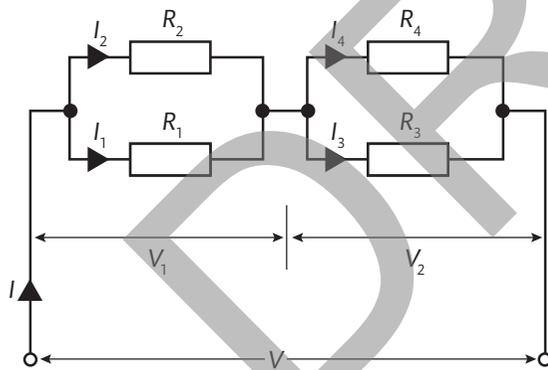
The approach is to split the circuit into smaller elements that can be treated as purely series or purely parallel circuits. Then apply the appropriate formulae and replace each series or parallel element with an equivalent single resistor.

The following worked example demonstrates the principles of simplifying the network to find a single equivalent resistance. The same method can be applied to analyse a network containing five or more resistors.

Worked Example

Consider the circuit shown in **Figure 1.81**, where $R_1 = 47\ \Omega$, $R_2 = 220\ \Omega$, $R_4 = 270\ \Omega$ and the supply voltage is 200 V. The total power dissipated by the circuit is 190 W.

Calculate the value of R_3 .



► **Figure 1.81** Resistors connected in a series and parallel combination

Solution

Determine the total circuit current using the voltage and power dissipation.

$$\text{From } P = IV, I = \frac{P}{V} = \frac{190}{200} = 0.95 \text{ A or } 950 \text{ mA}$$

You can find the total circuit resistance from $V = IR$:

$$R_{\text{TOTAL}} = \frac{V}{I} = \frac{200}{0.95} = 210.5\ \Omega$$

Now find the equivalent single resistor R_A that could replace the parallel network involving R_1 and R_2 .

$$\frac{1}{R_A} = \frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{47} + \frac{1}{220} = 0.0258$$

$$\text{so } R_A = \frac{1}{0.0258} = 38.8\ \Omega$$

You now know the total resistance R_{TOTAL} and the equivalent resistance of the R_1 and R_2 parallel network (R_A).

The R_3 and R_4 parallel network (R_B) is connected in series with R_A , so $R_{\text{TOTAL}} = R_A + R_B$ and therefore $R_B = R_{\text{TOTAL}} - R_A = 210.5 - 38.8 = 171.7\ \Omega$.

You know that

$$\frac{1}{R_B} = \frac{1}{R_3} + \frac{1}{R_4}$$

$$\text{so } \frac{1}{R_3} = \frac{1}{R_B} - \frac{1}{R_4} = \frac{1}{171.7} - \frac{1}{270} = 0.00212$$

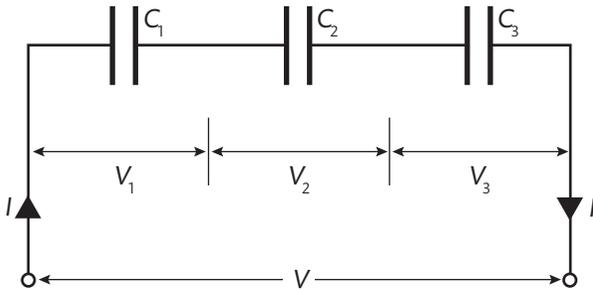
$$\text{so } R_3 = \frac{1}{0.00212}$$

$$R_3 = 472\ \Omega$$

Capacitors in series and parallel

Capacitors in series

Figure 1.82 shows three capacitors connected in series. In this configuration it is important to remember that the charge (q) on each capacitor will be the same.



► **Figure 1.82** Capacitors connected in series

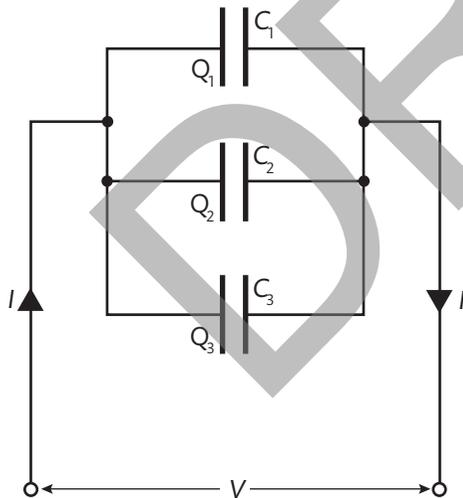
By applying Kirchhoff's laws, you can show that the total capacitance (C_T) for a number of capacitors (n) in series is:

$$\frac{1}{C_T} = \frac{1}{C_1} + \frac{1}{C_2} + \dots + \frac{1}{C_n}$$

Capacitors in parallel

Figure 1.83 shows three capacitors connected in parallel. By considering the total charge as being the sum of the charges stored in each capacitor, you can show that the total capacitance (C_T) for a number of resistors (n) in series is:

$$C_T = C_1 + C_2 + \dots + C_n$$



► **Figure 1.83** Capacitors connected in parallel

Direct current sources

Cells – these are electrochemical devices able to generate an e.m.f. and so can be used as a source of electrical current. Cells are generally limited by their chemistry to producing relatively low voltages; for example, a typical zinc-carbon cell produces a voltage of only 1.5V.

Battery – a simple battery can be made up of a single cell or multiple cells. Multiple cells connected in series provide a higher voltage than a single cell; if connected in parallel, they provide a higher current.

Internal resistance – this is the resistance of the internal structure of the battery or cell. It usually manifests itself as heat as the battery discharges.

Stabilised power supply – a device that produces an accurately regulated d.c. voltage output from an a.c. input. It often has a means of limiting the current in case of a circuit malfunction or dead short.

Photovoltaic (PV) cell – a semiconductor device that generates an e.m.f. when exposed to light. Individual cells are able to generate only small voltages and are usually connected together in arrays to enable them to provide more useful voltage or current output characteristics.

Worked Example

A battery with an e.m.f. of 12V and internal resistance $1\ \Omega$ is connected to a resistive load of $9\ \Omega$.

Calculate the voltage present at the battery terminals.

Solution

Calculate the circuit current using Ohm's law:

$$I = \frac{V}{R} = \frac{12}{9 + 1} = 1.2\text{ A}$$

The voltage across the $9\ \Omega$ load is therefore:

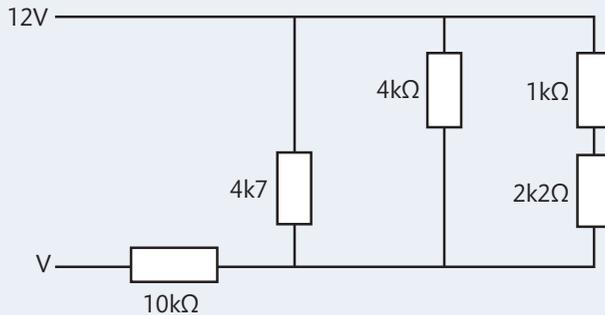
$$V = IR = 1.2 \times 9 = 10.8\text{ V}$$

By Kirchhoff's law, the voltage being supplied at the terminals of the battery must be 10.8V.

Assessment practice 1.5

- 1 The circuit diagram in **Figure 1.84** shows a 12V d.c. power source connected to a network of resistors.

Note: '4k7' means '4.7 kΩ'. This notation is commonly used in engineering to ensure that decimal points are not missed when printed on small component labels.

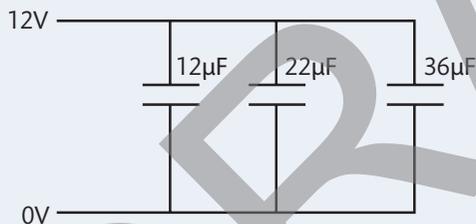


► **Figure 1.84** Network of resistors

Calculate the total current flowing in the circuit. (3 marks)

- 2 The electronic circuit diagram in **Figure 1.85** shows a d.c. power source connected to three capacitors connected in parallel.

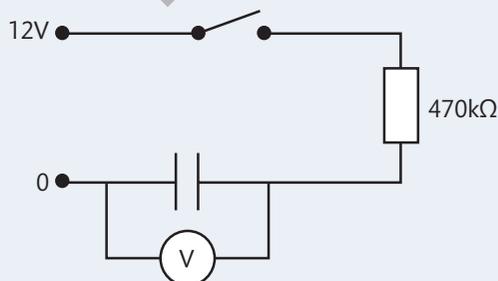
- Calculate the total equivalent capacitance of the three parallel capacitors. (1 mark)
- Calculate the charge stored in each capacitor. (3 marks)
- Calculate the total energy stored in the circuit. (1 mark)



► **Figure 1.85** A d.c. power source connected to three capacitors connected in parallel

- 3 A voltage of 12 V is connected to a fully discharged capacitor through a resistance of 470 kΩ, as shown in **Figure 1.86**. After 2 seconds the voltage across the capacitor is 7.72 V. The voltage (V_C) across a charging capacitor at any time (t) is given by the equation

$$V_C = V_0(1 - e^{-t/RC})$$



► **Figure 1.86** A capacitor-charging circuit

- Calculate the size of the capacitor. (6 marks)
- Calculate the time constant for the circuit. (1 mark)

For each of the problems in this assessment practice, use the following stages to guide your progress through the task.

Plan

- What are the important engineering concepts that I could apply to this problem?
- Do I feel confident that my understanding of this topic is sufficiently developed to solve the problem?

Do

- I can approach the task logically and methodically, clearly recording each step in my problem-solving process.
- I can identify when I have gone wrong and adjust my approach to get myself back on course.

Review

- I can effectively review and check mathematical calculations to prevent or correct errors.
- I can explain the best way to approach this type of problem.

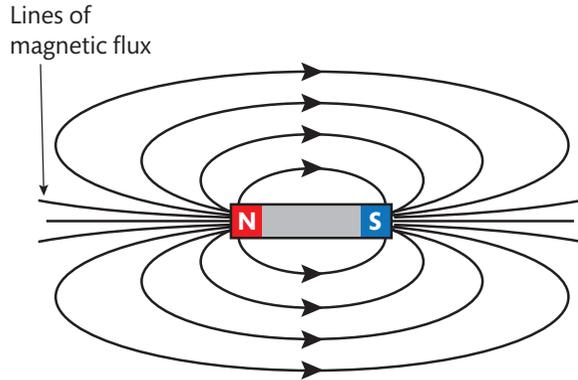
F

Magnetism and electromagnetic induction

F1 Magnetism

Magnetic fields

Magnetic fields are often represented by lines of magnetic flux that seem to flow around sources of magnetism such as a simple bar magnet (see **Figure 1.84**).



► **Figure 1.87** Lines of magnetic flux around a bar magnet

The following are important parameters that are used to describe magnetic fields:

- **Magnetic flux (Φ)** – a measure of the size of the magnetic field produced by a source of magnetism. Unit: weber (Wb).
- **Magnetic flux density (B)** – a measure of how closely packed the lines of magnetic flux (Φ) produced by a source of magnetism with area (A) are. It is used to represent the strength or intensity of a magnetic field. Units: webers per square metre (Wb m^{-2}) or tesla (T).
- **Magnetomotive force (m.m.f.) (F_m)** – the force that causes the formation of magnetic flux in a ferromagnetic material placed inside a solenoid or coiled conductor with (N) turns. It is generated by a current flow (I) through the solenoid. Unit: ampere (A).

$$B = \frac{\Phi}{A}$$

- **Magnetic field strength (H)** – a measure of the strength of the magnetising field produced by a solenoid with mean length (L) and number of turns (N) carrying a current (I). Unit: amperes per metre (A m^{-1}).

$$H = \frac{NI}{L}$$

- **Permeability (μ)** – a measure of the degree of magnetisation a material undergoes when subject to a magnetic field. In other words, it compares the magnetic field strength (H) with the magnetic flux

density (B) generated inside a material. Unit: henrys per metre (H m^{-1}).

$$\mu = \frac{B}{H}$$

- **Permeability of free space (μ_0)** – because there is no material to influence a magnetic field in a vacuum, the applied magnetising field strength (H) and flux density (B) will be the same. However, since you measure these two quantities in different units, in practice you must use the value

$$\mu_0 = 4\pi \times 10^{-7} \text{ H m}^{-1}$$

- **Relative permeability (μ_r)** – compares the permeability (μ) calculated for a given material to the permeability of free space (μ_0).

$$\mu_r = \frac{\mu}{\mu_0} \text{ so } \mu_0 \mu_r = \frac{B}{H}$$

Ferromagnetic materials

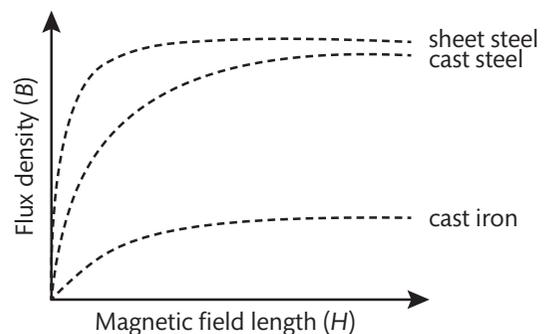
Ferromagnetic materials are strongly attracted by a magnetic field. They readily become magnets themselves under the influence of an external magnetic field and are able to retain their magnetic properties even after the external field has been removed.

B/H curves and loops

A **B/H curve** is a graphical representation of the relationship between the magnetic flux density (B) formed in a specific material when the material is exposed to a magnetic field strength (H).

B/H curves are not the straight lines that you might expect. They are curved because the permeability of ferromagnetic materials varies considerably with the strength of the applied magnetic field (H).

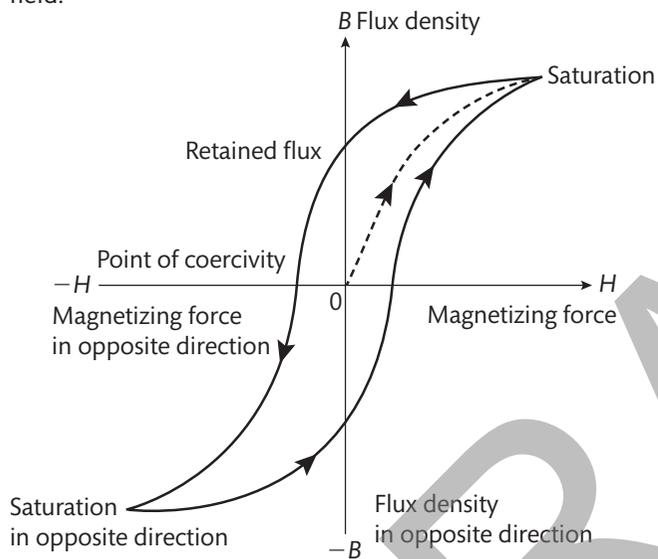
Figure 1.88 shows the B/H curves for some common ferromagnetic materials. Each curve starts at a point where B and H are zero and the material is entirely



► **Figure 1.88** Typical B/H curves for some ferromagnetic materials

demagnetised. In each case, you can see that as the magnetic field strength is increased, there is initially a rapid increase in flux density. The rate of increase in B then slows considerably as the curve flattens out. The material is said to have reached magnetic saturation when any further increase in H leads to a negligible increase in B .

You can gain a greater understanding of the behaviour of ferromagnetic materials by considering the effects of reversing the applied magnetic field once saturation has been reached. Consider the **B/H loop** shown in **Figure 1.89**. The dashed line shows a typical B/H curve, which illustrates how a completely demagnetised material becomes magnetised with the application of a magnetic field.



► **Figure 1.89** A typical B/H hysteresis loop

As you reduce the magnetic field (H) back to zero, the flux density (B) also decreases, but some retained flux remains in the material. When the applied magnetic field is reversed, this retained flux reduces back to zero. The size of the magnetic field required to eliminate the retained flux is known as the **coercivity** of the material.

As you build the reversed magnetic field further, magnetic saturation in the opposite direction is reached.

Reducing and then reversing the magnetic field once again has a similar effect on the material and so a closed loop is formed.

Hysteresis

Any change in flux density (B) lags behind changes in the applied magnetic field strength (H). This effect is called **magnetic hysteresis** and the loop described in **Figure 1.89** is known as the **hysteresis loop**.

The internal realignments that occur during the magnetisation cycle in a ferromagnetic material are responsible for energy lost as heat. This is called **hysteresis loss**. This energy loss is proportional to the area inside the hysteresis loop and can vary considerably across different materials.

In applications where materials undergo the magnetisation cycle many times a second, as in a.c. transformers, choosing a material with a narrow hysteresis loop is vital to prevent excessive losses and overheating.

Reluctance

Some metals exhibit a resistance to the presence of a magnetic field. The **reluctance** (S) is used as a measure of this magnetic resistance and can be expressed as the ratio of m.m.f. (F_m) to magnetic flux (Φ). Unit: inverse henry (H^{-1}).

$$S = \frac{F_m}{\Phi}$$

Reluctance can also be expressed in terms of the mean length of the flux path (l), the area through which the flux is flowing (A), the permeability of free space (μ_0) and the relative permeability of the material (μ_r) through which the field is passing:

$$S = \frac{l}{\mu_0 \mu_r A}$$

Magnetic screening

Often, it is necessary to screen components or devices from magnetic fields to prevent any unwanted effects. This is a common requirement when dealing with sensitive electronic devices.

Effective screening can be achieved using ferromagnetic materials with low reluctance, which can provide a pathway for lines of magnetic flux around the object or objects being protected.



PAUSE POINT

Explain what is meant by the term magnetic hysteresis.

Hint

Draw a graph to help illustrate your explanation.

Extend

What is the significance of the area inside a hysteresis loop? What possible consequences could this have for the design of components that rely on a continuous magnetisation cycle for their operation, such as a transformer?

F2 Electromagnetic induction

Induced electromotive force

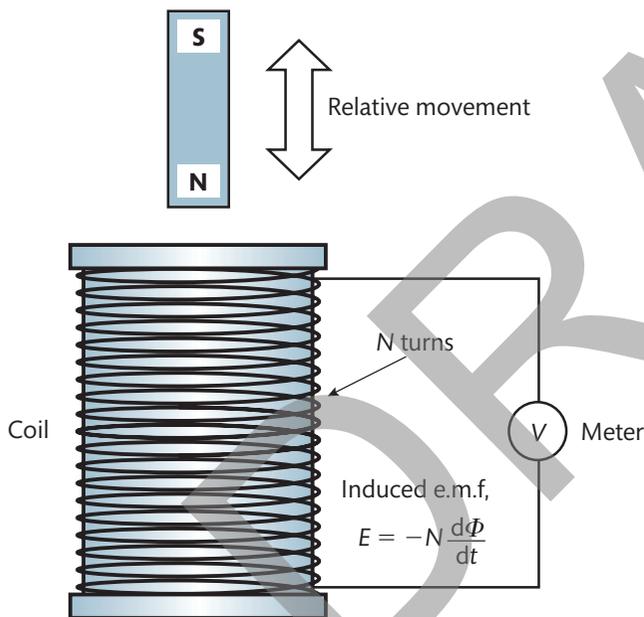
Electromagnetic induction describes the phenomenon by which an electromotive force (e.m.f.) is induced or generated in a conductor when it is subjected to a changing magnetic field according to **Faraday's laws of electromagnetic induction**.

Key terms

Faraday's laws of induction – combined, these laws state: 'When a magnetic flux through a coil is made to vary, an e.m.f. is induced. The magnitude of this e.m.f. is proportional to the rate of change of flux.'

This can be readily demonstrated by moving a permanent magnet through a coil of wire, as shown in **Figure 1.90**.

Lenz's Law – an induced current always acts in such a direction so as to oppose the change in flux producing the current.



► **Figure 1.90** Demonstration of electromagnetic induction using a coil and a bar magnet

An e.m.f. will be generated in the coil whenever the magnet is moved into or out of the coil and perpendicular lines of magnetic flux cut through the conductor. (Alternatively, the magnet could remain still and the coil could be moved.)

The size of the induced e.m.f. (E) depends on the number of turns in the coil (N) and the rate of change of magnetic flux ($\frac{d\Phi}{dt}$):

$$E = -N \frac{d\Phi}{dt}$$

The negative sign is a consequence of **Lenz's law**, which states that the induced current will act to oppose the change in flux.

An alternative approach to determine the e.m.f. (E) considers the rate at which lines of flux are being cut by a conductor of length (L) travelling at velocity (v) through a magnetic field with flux density (B).

$$E = BLv$$

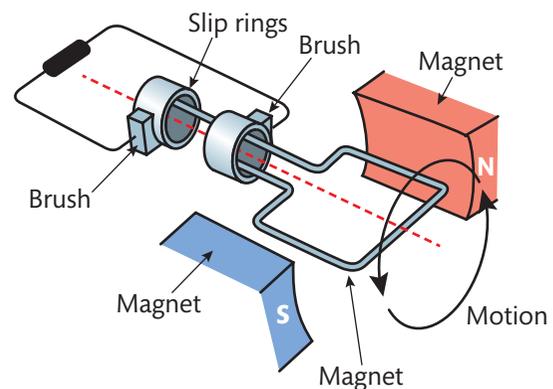
Note that this relationship assumes that the conductor is moving at right angles to the lines of flux.

Eddy currents

As well as inducing an e.m.f., which is able to flow as useful current through the conductor, other localised currents (eddy currents) are also created inside the conductor. These obey Lenz's law by flowing in a direction opposite to the changing magnetic field that created them. As a result, a proportion of useful energy is lost in the form of heat. Devices in which eddy currents occur, such as transformers, have to be carefully designed to prevent excessive energy loss and overheating. For example, transformers are constructed with laminated thin soft iron plates, which are electrically isolated from one another using a thin layer of lacquer to restrict the size of the eddy currents that form.

Electric generators

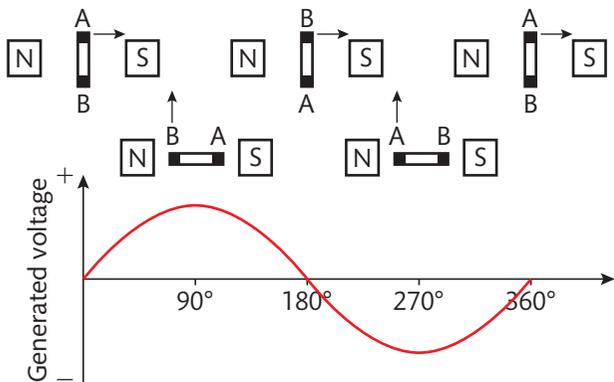
Electric generators are usually based on the principle of rotating a coiled conductor inside a static magnetic field (see **Figure 1.91**). The ends of the coil are connected to copper slip-rings that maintain contact with stationary carbon brushes through which the induced current flows.



► **Figure 1.91** A simple electric generator

The magnitude of the e.m.f. induced in the coil is proportional to the number of turns of the coil. It is also dependent on the speed of rotation and the strength of the magnetic field.

The voltage generated by a rotating coil is sinusoidal because the angle between the coil and the lines of flux varies according to the angle of rotation (see **Figure 1.92**). Peak voltage is achieved where the motion of the coil is perpendicular to the lines of flux, that is, cutting through the flux lines at right angles. Voltage falls to zero when the coil has rotated through 90° because at this stage the motion of the coil becomes parallel to the lines of flux and so does not cut through them at all.



► **Figure 1.92** The sinusoidal voltage generated by a rotating coil

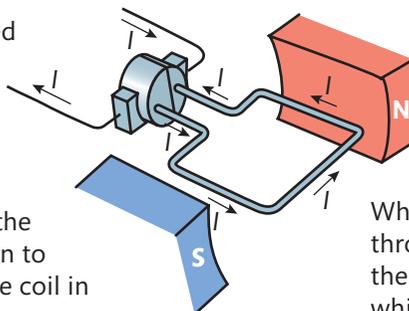
Direct current electric motors

An electric motor (see **Figure 1.93**) uses similar principles to an electric generator, but instead produces movement when a current is passed through a coil that is free to rotate inside a magnetic field.

When a current is passed through the coil, its associated magnetic field opposes the field generated by the permanent magnets and causes the coil to rotate until the two fields are aligned. After rotating 180°, the split slip-ring (the commutator), which provides current to the rotating coil via carbon brushes, reverses the current flow in the coil and hence the direction of its magnetic field. Once again the magnetic field generated by the coil opposes the field generated by the permanent magnets, and rotation continues.

Electric current supplied externally through a commutator

The commutator reverses the current each half revolution to keep the torque turning the coil in the same direction



When electric current (I) passes through a coil in a magnetic field, the magnetic force produces a torque which turns the d.c. motor

► **Figure 1.93** Basic d.c. electric motor

In practice, the rotating element (the armature) is mounted on bearings to allow free rotation and to minimise frictional losses. The armature encompasses the main drive shaft of the motor, a laminated iron core, around which several individual coils are wound, and the commutator. The commutator is split into several segments, with each feeding different coils (or windings) in turn as the armature rotates. This arrangement helps to ensure smooth rotation and torque delivery.

Also, large motors do not use permanent magnets in the stationary part of the motor (the stator). Instead, the stator consists of a laminated iron ring with further windings. These field windings are arranged so as to provide the necessary stationary magnetic field when current flows through them.

Self-inductance in a coil

According to Lenz's law, any change in the current flowing in a circuit will generate a changing magnetic field, which in turn induces a back e.m.f. that opposes the current change.

Self-inductance describes a situation where an e.m.f. is induced in the same circuit in which the current is changing.

Self-inductance (L) is measured in henrys (H), where 1 H is the inductance present in a circuit where a changing current of 1 A s^{-1} induces an e.m.f. of 1 V. The induced e.m.f. (E) is therefore related to the rate at which the current changes ($\frac{dI}{dt}$) and the inductance of the circuit (L):

$$E = -L \frac{dI}{dt}$$

You already know that the e.m.f. induced in a coil can also be calculated from the rate of change of magnetic flux present:

$$E = -N \frac{d\Phi}{dt}$$

These two relationships allow you to calculate the self-inductance (L) using the number of turns in the coil (N), the current (I) and the generating magnetic flux (Φ):

$$L = \frac{N\Phi}{I}$$

II PAUSE POINT

Use notes and an annotated diagram to explain the operation of a d.c. motor.

Hint

A good diagram will often convey much of the necessary information.

Extend

Why do industrial electric motors use field windings instead of permanent magnets in their construction and operation?

Energy storage in an inductor

When you first establish the flow of current in an inductive circuit, a back e.m.f. is induced to oppose the increasing current flow. As a consequence, additional energy is required to establish the current flow. This additional energy is then stored within the magnetic field generated by the current and is recovered when the current stops and the magnetic field collapses.

The energy stored in an inductor (W) is the product of the circuit inductance (L) and the current (I). Unit: joule (J).

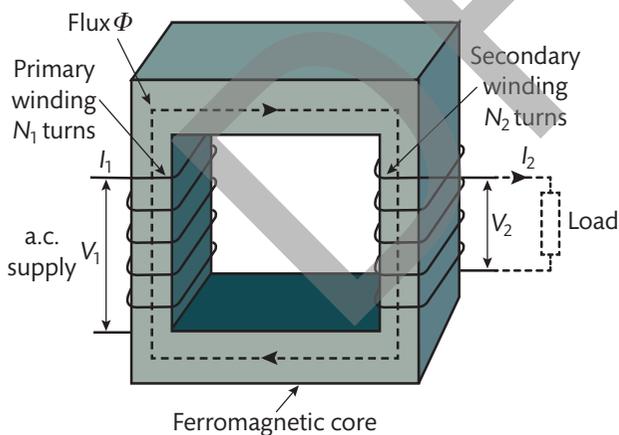
$$W = \frac{1}{2}LI^2$$

Mutual inductance

Mutual inductance (M) describes a situation where a changing current in one circuit induces an e.m.f. in an adjacent circuit. It is measured in henrys (H), where 1 H is the mutual inductance present where a changing current of 1 A s^{-1} in one circuit induces an e.m.f. of 1 V in the second.

Transformers

The most common and important application of mutual inductance is in transformers (see **Figure 1.94**).



► **Figure 1.94** A simple transformer

A transformer consists of two separate coils or windings wound on a common ferromagnetic core (commonly constructed from laminated steel to minimise losses caused by eddy currents). The primary winding is connected to an a.c. electrical supply and the other, secondary, winding is connected to an electrical load.

The varying current in the primary winding generates magnetic flux in the transformer core. The flux flows around the core in a magnetic circuit and induces an e.m.f. in both the primary and the secondary windings. If we assume an ideal transformer (one which is well designed with negligible losses), then the rate of change of flux will be the same for each winding, and so the induced e.m.f. will be dependent on the number of turns in each winding.

Where the primary coil has number of turns (N_1) and voltage (V_1) and the secondary coil has number of turns (N_2) and voltage (V_2), the transformer voltage ratio is given by

$$\frac{V_1}{V_2} = \frac{N_1}{N_2}$$

► In a **step-up transformer**, the voltage ratio $\frac{V_1}{V_2} < 1$, that is, the secondary voltage is greater than the primary voltage.

► In a **step down transformer**, the voltage ratio $\frac{V_1}{V_2} > 1$, that is, the secondary voltage is less than the primary voltage.

► In an **ideal transformer**, the laws of conservation of energy mean that the power in the primary coil is the same as the power in the secondary coil:

$$V_1 I_1 = V_2 I_2$$

The transformer current ratio is then given by

$$\frac{V_1}{V_2} = \frac{I_2}{I_1}$$

II PAUSE POINT

Explain two significant sources of losses that will affect the efficiency of a transformer.

Hint

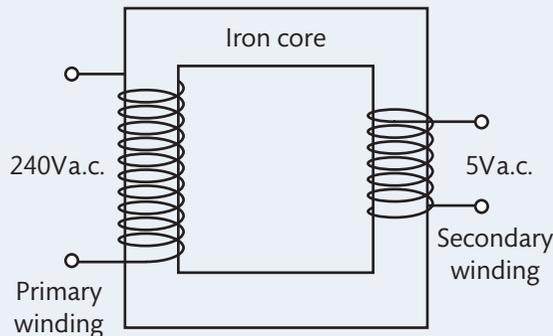
Most losses occur within the transformer core.

Extend

Explain how a transformer core is designed to minimise these losses.

Assessment practice 1.6

- 1 The diagram in **Figure 1.95** shows a transformer used in a phone charger that reduces mains voltage at 230V a.c. to 5V a.c. There are 800 turns of wire on the primary winding.



▶ **Figure 1.95** A transformer in a phone charger

- a) Calculate the number of turns required in the secondary coil. (2 marks)
- b) The 5V a.c. must be converted to 5V d.c. before it can be used to power the phone. Explain what is meant by full-wave rectification. (2 marks)
- c) Draw a circuit diagram for a full-wave rectifier circuit. (3 marks)
- 2 Calculate the energy stored in an inductor with inductance 0.8 H passing a current of 2.6 A. (1 mark)

For each of the problems in this assessment practice, use the following stages to guide your progress through the task.

Plan

- Have I extracted and summarised all the information available in the question?
- How confident do I feel that I have the knowledge required to approach this task?

Do

- I have spent adequate time planning my approach. I can clearly explain the steps involved and the order in which they should be done.
- I can recognise when my method is leading nowhere and I need to stop and rethink my approach.

Review

- I can appreciate the importance of ensuring that my solution is laid out and explained with sufficient clarity so that it can be followed by someone else.
- I can explain the improvements I would make to my approach the next time I encounter a similar problem.

G

Single-phase alternating current

Single-phase alternating current theory

Alternating current (a.c.) electricity is usually generated by means of a coil rotating within a magnetic field. For one revolution of the coil, the resulting e.m.f. will alternate between a maximum positive and maximum negative value. This type of electrical generator is also known as an alternator.

Sinusoidal waveforms

When the value of the e.m.f. generated by an alternator is plotted against time, the resulting waveform is sinusoidal. Just like the pure sine functions (see 'Graphs of the trigonometric functions' in **A2 Trigonometric methods**), a sinusoidal waveform is periodic – it is a series of identical repeating cycles.

Two important parameters are used to describe all periodic waveforms.

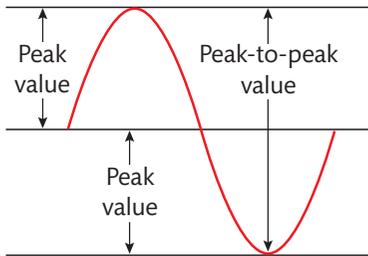
- ▶ **Periodic time** (T) – the time taken to complete one cycle, also called simply the **period**. Unit: second (s).
- ▶ **Frequency** (f) – the number of cycles completed in one second. Unit: hertz (Hz).

The periodic time (T) and the frequency (f) are the reciprocal of each other:

$$T = \frac{1}{f} \text{ and } f = \frac{1}{T}$$

For a sinusoidal a.c. current, there are a variety of measures that can be used to describe the characteristics of the waveform.

- ▶ **Peak** – the maximum value of voltage or current reached in a positive or negative half-cycle (see **Figure 1.96**).
- ▶ **Peak-to-peak** – the difference between the positive peak and the negative peak voltage or current in a full cycle (see **Figure 1.96**).



- ▶ **Figure 1.96** Peak and peak-to-peak values of an a.c. signal
- ▶ **Root mean square (r.m.s.)** – the value of a direct current that would produce an equivalent heating effect as the alternating current. For a sine wave, the r.m.s. voltage (V_{rms}) is related to the peak voltage (V_{peak}) by

$$V_{rms} = \frac{1}{\sqrt{2}} V_{peak} \quad \text{or, in terms of current: } I_{rms} = \frac{1}{\sqrt{2}} I_{peak}$$

- ▶ **Average** – the average of all the instantaneous measurements in one half-cycle. For a sine wave, the average voltage (V_{avg}) is related to the peak voltage (V_{peak}) by

$$V_{avg} = \frac{2}{\pi} V_{peak} \quad \text{or, in terms of current: } I_{avg} = \frac{2}{\pi} I_{peak}$$

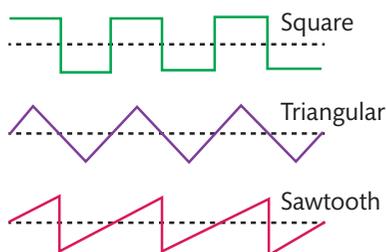
- ▶ **Instantaneous** – the value of voltage or current at a particular time instant during the sinusoidal cycle. Instantaneous voltage and current are zero when the waveform crosses the time axis where it changes polarity.
- ▶ **Form factor** – equal to the r.m.s. voltage (V_{rms}) divided by the average voltage (V_{avg}). For a sine wave, the form factor is a constant:

$$\text{form factor} = \frac{V_{rms}}{V_{avg}} = \frac{V_{peak}}{\sqrt{2}} \times \frac{\pi}{2V_{peak}} = \frac{\pi}{2\sqrt{2}} = 1.11$$

This relationship is also true for current.

Non-sinusoidal waveforms

Figure 1.97 shows square, triangular and sawtooth waves, which are all examples of non-sinusoidal waveforms.



- ▶ **Figure 1.97** Some examples of non-sinusoidal waveforms

Case study

UK mains electrical supply

The majority of electricity in the UK is still generated by burning fossil fuels to drive steam turbines, which in turn drive generators. A two-pole generator supplies one complete a.c. cycle per rotation and so, when driven at 3000 rpm, it will generate 3000 cycles per minute or 50 cycles per second. It is extremely important for generating companies to control the rotational speed of their generators to maintain this 50 Hz frequency.

Domestic premises in the UK are supplied electricity through an extensive network of electrical supply cables and infrastructure, collectively known as the national grid. A series of local substations receive voltages, typically 11 kV 50 Hz a.c., from overhead supply lines and then use step-down transformers to output the 230V 50 Hz a.c. standard UK domestic supply voltage. This is then fed to homes through underground cables.

Check your knowledge

- 1 A substation transformer has 4500 windings on its primary coil. Calculate the number of windings required on the secondary coil.
- 2 Explain two significant causes of energy loss that can occur in transformers and how these can be minimised.
- 3 Calculate the peak voltage of a 230V a.c. sinusoidal waveform.
- 4 A 2.5A current flows through an inductor with inductance 0.3 H and a resistance of 9Ω when connected to a 240V 50 Hz a.c. supply. Calculate the change in current that would occur if the mains frequency were to drop to 45 Hz.

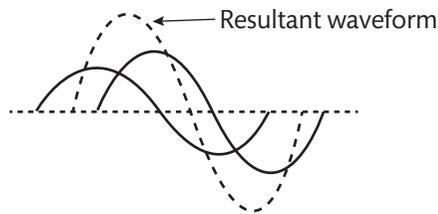
Combining sinusoidal waveforms

The result of adding two sinusoidal voltages together can be determined either graphically or by the vector addition of phasors.

Graphical approach

In **Figure 1.98**, two waveforms are drawn on the same axes. At any point on the time axis, the resultant instantaneous voltage is obtained by adding together the individual instantaneous voltages at that point. The dotted line shows the resultant waveform generated in this way.

This approach can be used to combine any kinds of waveforms, irrespective of their frequency, amplitude or phase difference. The obvious downside to this approach, as with all manual graphical techniques, is that it is time-consuming and the accuracy of the results will be limited.



► **Figure 1.98** Graphical representation of adding two waveforms

Vector addition of phasors

This approach builds on the techniques explained in the sections on phasors and vector addition in **A2 Trigonometric methods**. It should be noted that it is valid only when applied to sinusoidal waveforms with the **same frequency**.

Consider two a.c. waveforms represented by

$$v_1 = 100 \sin(100\omega t) \text{ and } v_2 = 200 \sin\left(100\omega t - \frac{\pi}{6}\right)$$

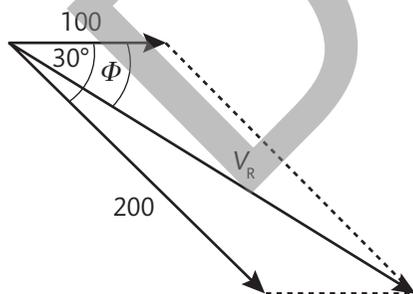
If you compare these to the standard form for a sinusoidal phasor

$$v = V \sin(\omega t + \Phi)$$

you can see that the magnitude and direction of the phasors are as follows:

- v_1 has magnitude 100 with no phase shift, so the direction of the phasor will be 0° (horizontal).
- v_2 has magnitude 200 with a phase shift of $-\frac{\pi}{6}$ rad, which means the direction of the phasor will be 30° below the horizontal.

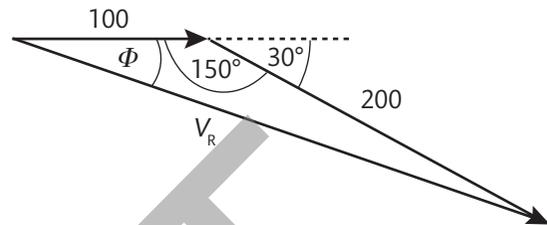
These phasors can therefore be drawn as in **Figure 1.99**.



► **Figure 1.99** Combining the phasors of a.c. waveforms

An accurate scale drawing will allow you to determine the magnitude of the resultant peak voltage (V_R) and phase angle (Φ) by direct measurement. It is often more convenient to convert the phase angle from radians to degrees to allow the use of a conventional protractor.

You could take a more analytical approach by redrawing the phasor diagram as shown in **Figure 1.100**.



► **Figure 1.100** Vector addition of phasors

The magnitude of the resultant peak voltage (V_R) can be determined from the cosine rule:

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\text{So } V_R^2 = 200^2 + 100^2 - 2 \times 200 \times 100 \times \cos 150^\circ = 84641$$

$$V_R = 290.93 \text{ V}$$

You can find the resultant phase angle (Φ) by using the sine rule:

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{290.93}{\sin 150^\circ} = \frac{200}{\sin \Phi}$$

Rearranging and solving for Φ gives

$$\Phi = 20.10^\circ \text{ or } 0.351 \text{ rad}$$

Then you can express the resultant waveform as

$$v_R \approx 291 \sin(\omega t - 0.35)$$

Remember that the phase angle must be in radians if it is to be used in the general form for a sinusoidal phasor.

Note that here the phase angle lags below the horizontal and so is considered negative.

Impedance

A.c. circuits that contain capacitors and/or inductors are more complex to analyse than those containing purely ohmic resistance.

II PAUSE POINT

Explain the application of phasors in the addition of a.c. waveforms.

Hint

You might find it helpful to review previous work on phasors in the **A2 Trigonometric methods** section.

Extend

Use notes and sketches to explain the relationship between a rotating vector represented by a phasor and a sinusoidal waveform.

Impedance (Z) is the total opposition to the flow of electricity exhibited in an a.c. circuit.

The voltage and current in a purely resistive a.c. circuit are in phase, and the impedance of the circuit is equal to its resistance.

In circuits containing either capacitance or inductance, the impedance is made up of two parts: the **resistance** (R), which is independent of frequency, and the **reactance** (X), which varies with frequency. There are two forms of reactance, both dependent on frequency (f):

- ▶ **Capacitive reactance** $X_C = \frac{1}{2\pi fC}$, where C is the capacitance value in farads. Unit: ohm (Ω).
- ▶ **Inductive reactance** $X_L = 2\pi fL$, where L is the inductance value in henrys. Unit: ohm (Ω).

The total impedance (Z) is made up of the resistance (R) and the reactance (X). However, these cannot simply be added together, as phase changes between voltage and current waveforms must be taken into account. Impedance is therefore the vector sum of the ohmic resistance and the reactance present in a circuit.

- ▶ For a circuit with resistance R and inductive reactance X_L , the total impedance (Z) is given by

$$Z = \sqrt{X_L^2 + R^2}$$

- ▶ For a circuit with resistance R and capacitive reactance X_C , the total impedance (Z) is given by

$$Z = \sqrt{X_C^2 + R^2}$$

Alternating current rectification

Rectification is the conversion of an a.c. waveform into d.c. You looked at this in the section on diodes in **E2 Direct current circuit theory**.

Assessment practice 1.7

- 1 Two a.c. voltage waveforms are represented by $v_1 = 65 \sin(80\omega t)$ and $v_2 = 90 \sin(80\omega t - \frac{\pi}{5})$.
 - a) Draw a sketch of the phasors that can be used to represent these waveforms. (2 marks)
 - b) Use trigonometric methods to find the magnitude and phase angle of the resultant phasor when v_1 and v_2 are combined. (4 marks)
 - c) State the resultant waveform in the standard form $v = V \sin(\omega t + \phi)$. (1 mark)
 - d) Calculate the frequency of the resultant waveform. (1 mark)
- 2 An engineer is testing an inductor with an inductance of 0.8 H and a resistance of 3 Ω . It is connected to a 240 V 50 Hz a.c. supply. Calculate the current drawn from the supply. (3 marks)

For each of the problems in this assessment practice, use the following stages to guide your progress through the task.

Plan

- Will a sketch help me to visualise the scenario given in the question?
- Is it clear to me how to approach this type of question? Will I have to refer back to my notes?

Do

- I can apply the methods taught in this unit to solve problems appropriately.
- I understand my own limitations and when I need to stop, revisit the textbook, look at my lesson notes or ask for help.

Review

- I can explain to others why I chose a particular method for solving this type of problem and how to apply it correctly.
- I can evaluate how effective my planning was in approaching this question and how I might make improvements for next time.

Assessment practice 1.8 synoptic question

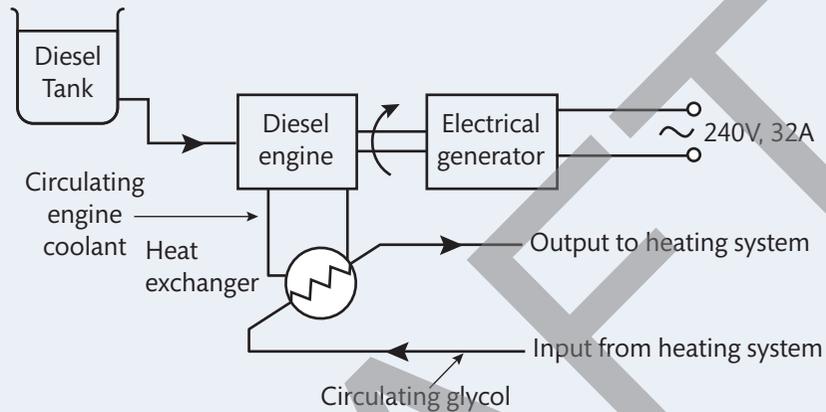
A remote ski cabin relies on a diesel engine to generate its electricity. The system is depicted in **Figure 1.101**.

The diesel used to run the engine has a total energy content of 36 MJ l^{-1} and is consumed at a rate of 3.2 l h^{-1} .

The generator driven by the diesel engine provides a 240 V electrical supply at 32 A .

Waste heat from the diesel engine is used to heat the cabin. This is transferred from circulating engine coolant via a heat exchanger into a glycol-filled heating system.

The diesel engine generates 15 kW of energy wasted as heat. The glycol in the heat circulation system has a specific heat capacity of $4.18 \text{ kJ kg}^{-1} \text{ K}^{-1}$ and a density of 1096 kg m^{-3} . The flow rate of glycol through the heat exchanger is 12 l s^{-1} and the temperature difference maintained between the input and output pipes is $+0.2^\circ \text{C}$.



► **Figure 1.101** Electricity generator using a diesel engine

- Calculate the thermal efficiency of the heat transfer between the diesel engine and the glycol heat circulation system. (7m arks)
- Calculate the overall system efficiency. (7m arks)

THINK ▶ FUTURE



Rob Taylor

Product designer
for an industrial
lighting company

I've been working as an industrial product designer as part of a multi-disciplinary design team for four years. In the industrial lighting products that we work on, function, reliability, safety and efficiency are all key factors that we must get right. Of course, aesthetics are also an important aspect of our work, especially when working with architects to design bespoke lighting fixtures for big projects.

I am lucky to work with colleagues who have a diverse range of electromechanical and electronic specialist knowledge and experience, and I am still learning new things every day. When designing products, I have access to a range of materials and manufacturing techniques in our UK factory. A single product might involve designing sheet metal parts, aluminium castings and plastic injection moulded components. Internal electrical wiring must then be specified and lamps and their electronic drive gear positioned inside the luminaire. Safe and reliable ways of suspending the complete luminaires from a high warehouse roof or integrating them as part of an office ceiling system also need to be considered. I have even worked to design curved aluminium reflectors to distribute light evenly throughout a work area. Every project brings something new, and with recent developments in high-output LED technology, which is revolutionising lighting, these are exciting times to be involved in the lighting industry.

Focusing your skills

Seeing the whole picture

When designing a complete electromechanical product, machine or system it is important to appreciate the principles that underpin its operation and how these can affect one another. Here are some example questions that a designer might ask:

- Does the wiring size specified have a current rating significantly above the levels expected during normal operation? What would happen if safe working loads were exceeded?
- What would be the effects of extreme high or low temperatures on electrical and mechanical systems?
- Are mechanical or electrical systems likely to lose energy as heat during operation? What might the main losses be and how could these be reduced?
- If cooling is required, how much heat energy would need to be removed? Which type of coolant would be appropriate, what flow rate would be needed, and how could the coolant be circulated?
- Are electrical devices likely to cause electromagnetic interference and affect the operation of nearby sensitive electronic systems? If there is no room for the devices to simply be moved away from each other, how else might such interference be prevented?
- Are all the components of sufficient mechanical strength to withstand the working stresses encountered during operational extremes?
- How can I minimise the amount of material used to achieve the required outcome? What solid shapes might be most useful?
- How will the static forces existing in mechanical components be affected by linear or rotational movement?

Getting ready for assessment

This section has been written to help you to do your best when you take the external examination. Read through it carefully and ask your tutor if there is anything you are not sure about.

About the test

This unit is externally assessed using an unseen paper-based examination. Pearson sets and marks the examination. The assessment must be taken under examination conditions.

The assessment test will last two hours and there are a maximum of 80 marks available.

The test paper is divided into three sections (A, B and C).

Section A: Applied Mathematics has a series of short-answer questions each worth up to 2 marks. There are a total of 10 marks available in Section A.

Section B: Mechanical and Electrical/Electronic Principles has a series of short- and longer-answer questions each worth up to 7 marks. There are a total of 56 marks available in Section B.

Section C: Synoptic Question has a single long-answer question worth 14 marks.

Marks quoted for each section are based on Sample Assessment Materials.

Remember that all the questions are compulsory and you should attempt to answer each one.

Organise your time based on the marks available for each question. Set yourself a timetable for working through the test and then stick to it – do not spend ages on a short 1–2 mark question and then find you only have a few minutes for a longer 6–7 mark question.

Try answering all the simpler questions first, then come back to the harder questions. This should give you more time for the harder questions.

Sitting the test

Listen to and read carefully any instructions you are given. Lots of marks are often lost through not reading questions properly and misunderstanding what the question is asking.

Most questions contain command words. Understanding what these words mean will help you to understand what the question is asking you to do.

Command word	Definition – what it is asking you to do
Calculate	Learners judge the number or amount of something by using the information they already have, and add, subtract, multiply or divide numbers. For example, 'Calculate the reaction forces...'
Describe	Learners give a clear, objective account in their own words showing recall, and in some cases application, of the relevant features and information about a subject. For example, 'Describe the process of heat transfer...'
Draw	Learners make a graphical representation of data by hand (as in a diagram). For example, 'Draw a diagram to represent...'
Explain	Learners make something clear or easy to understand by describing or giving information about it. For example, 'Explain one factor affecting...'
Find	Learners discover the facts or truth about something. For example, 'Find the coordinates where...'
Identify	Learners recognise or establish as being a particular person or thing; verify the identity of. For example, 'Identify the energy loss...'
Label	Learners affix a label to; mark with a label. For example, 'Label the diagram to show...'
Solve	Learners find the answer or explanation to a problem. For example, 'Solve the equation to...'
State	Learners declare definitely or specifically. For example, 'State all three conditions for...'

Sample answers

For some of the questions you will be given background information on which the questions are based.

Look at the sample questions that follow and our tips on how to answer these well.

Answering applied mathematics calculation questions

- Identify the method required to solve the type of problem set.
- Explain each step and show all your workings.
- Express your answer in appropriate units.
- Check your answer.

Worked example

The velocity of a model rocket fired vertically upwards is given by the equation $v = 2t^2 + 5t - 11$.

Find, by use of the quadratic formula, the time when the rocket reached its highest point. (2 marks)

Answer

At its highest point the velocity of the rocket will have fallen to 0, so

$$2t^2 + 5t - 11 = 0$$

The quadratic formula takes the form

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In this problem:

$$a = 2, b = 5, c = -11$$

$$\text{so } t = \frac{-5 \pm \sqrt{5^2 - 4 \times 2 \times (-11)}}{2 \times 2} = \frac{-5 \pm 10.63}{4} = 1.41 \text{ or } -3.91$$

The negative value is not valid in the context of the question, so check the answer by substituting $t = 1.41$ into $v = 2t^2 + 5t - 11$:

$$v = 3.96... + 7.03... - 11 = 0 \text{ (remember to use unrounded values)}$$

The highest point will be reached after 1.41 s.

Where calculations are involved, marks are awarded for 'knowing a method and attempting to apply it' at each stage. It is essential, therefore, to show your working and make clear which principle or formula is being applied. Further marks are awarded for accuracy, that is, for reaching the correct numerical answer, only if the relevant method marks have already been awarded.

Answering engineering principles calculation questions

- Identify the variables from the information given in the question.
- Write down the formula you need.
- Substitute numbers into the formula.
- Explain each step and show all your workings.
- Express your answer in appropriate units.
- Check your answer.

Worked example

An engineer is testing an inductor with an inductance of 0.5 H and a resistance of 8 Ω . It is connected to a 120 V 50 Hz a.c. supply.

Calculate the current drawn from the power supply. (3 marks)

Answer

The solution will require the calculation of I_{rms} .

The values given in the question are:

$$L = 0.5 \text{ H}, R = 8 \Omega, V_{\text{rms}} = 120 \text{ V}, f = 50 \text{ Hz}$$

The inductive reactance is $X_L = 2\pi fL = 2\pi \times 50 \times 0.5 = 157.08 \Omega$.

The total impedance is given by $Z = \sqrt{X_L^2 + R^2} = \sqrt{157.08^2 + 8^2} = 157.28 \Omega$.

So the a.c. current is $I_{\text{rms}} = \frac{V_{\text{rms}}}{Z} = \frac{120}{157.28} = 0.763 \text{ A}$ or 763 mA.

Answering engineering principles short-answer questions – state

- Read the question carefully.
- Make sure that you make the same number of points as there are marks available in the question.

Worked example

State two sources of energy loss that affect the efficiency of electrical transformers. (2 marks)

Answer

Eddy currents. Hysteresis losses.

This answer provides two accurate points and full marks would be awarded. Command words such as 'give', 'state' or 'identify' can be answered in single words or brief statements. There is no need to write in full sentences because the examiner is only testing your ability to recall information.

Answering engineering principles short-answer questions – describe, explain

Worked example

Describe the process of heat transfer through conduction. (4 marks)

Answer

Heat transfer by direct contact between adjacent atoms:

Heat energy is a measure of the vibrational kinetic energy possessed by atoms. These vibrations are passed between atoms that are in direct contact with each other.

Heat transfer by the movement of electrons:

Heat energy can also be distributed through a material by the transfer of vibrational kinetic energy possessed by free-moving electrons as they move around within a material. This helps to explain why metals are in general good thermal (as well as electrical) conductors.

The mechanism of energy transfer through direct contact between atoms and the secondary mechanism involving energy transfer by free-moving electrons have been correctly identified. Each mechanism is then explained and expanded upon sufficiently for the award of the additional marks available.

Answering synoptic questions

Worked example

A heat engine driving a combined heat and power system outputs mechanical energy to drive a generator.

The energy is provided by a combustion process that uses air and a fuel with an energy content of 40 MJ kg^{-1} , which is supplied at a rate of 0.003 kg s^{-1} . The generator has a rotor that turns at 1500 rpm and has a torque of 255 N m. The output of the generator is 6.5 A at 400 V.

- Explain how energy loss processes in both mechanical and electrical equipment affect the efficiency of the system. (4 marks)
- Calculate the efficiency with which the heat engine provides mechanical work to the generator. (7 marks)
- Calculate the overall system efficiency. (3 marks)

Synoptic questions, such as the example given above, are basically a combination of interrelated short-answer and calculation questions based around a common complex system or theme. As such they can be broken down into individual elements that should be approached in a similar way to the worked examples already discussed.

Synoptic questions will require you to make links between engineering principles from the full range of disciplines covered in the unit.